Phys 312 2021

Lecture 2

* HW 1 by 5pm

* HW 2 by 5pm, Tuesday.

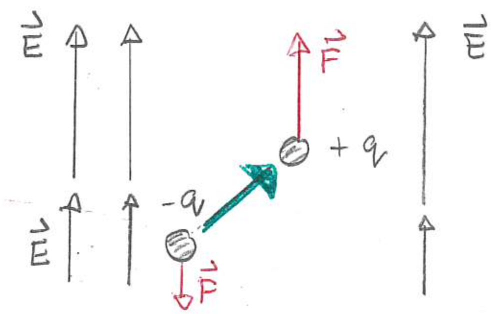
Tues: Read 4.3, 4.4.1

Electric Dipoles in External Fields

We will model materials as collections of dipoles and we need to consider how these will respond to external electric fields. We consider first a dipole consisting of two point charges. Suppose that this is placed in an external electric field pointing in one direction everywhere

The field exerts forces on each charge given by $\vec{F} = q\vec{E}$

These are as illustrated.



We can see:

1) the net force on the dipole will be zero if the field is uniform. The net force will be non-zero if the field is non-uniform or has any type of gradient.

2) the net torque depends on how the dipole moment aligns with the field. If $\vec{p}$ is along or opposite to $\vec{E}$ then the net torque is zero. If it is aligned in any other way it is non-zero.

We need quantitative relationships for these. These will be provided by the basic rules for electrostatics.

We obtain:

Consider a charge distribution whose net charge is zero, and which is placed in an external electric field, $\vec{E}_{ext}$. In the dipole approximation the net force on the distribution is

$$\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E}_{ext}$$

and the net torque in a uniform field is:

$$\vec{\tau} = \vec{p} \times \vec{E}_{ext}.$$

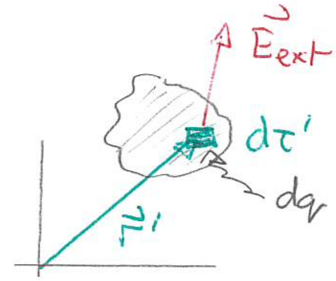
Here
$$\vec{p} \cdot \vec{\nabla} = p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z}$$

Proof:

Net force

The Lorentz force law for a point charge $\vec{F} = q\vec{E}_{ext}$ can be adapted to a distribution and gives

$$d\vec{F} = dq\, \vec{E}_{ext}$$
$$= \rho(\vec{r}\,')d\tau'\, \vec{E}_{ext}(\vec{r}\,')$$



Thus

$$\vec{F} = \int_{\text{all space}} \rho(\vec{r}\,')\, \vec{E}_{ext}(\vec{r}\,')\, d\tau'$$

We now expand $\vec{E}_{ext}(\vec{r}\,')$ using a general version of Taylor's theorem

For any function $g(\vec{r}\,')$

$$g(\vec{r}\,') = g(x', y', z')$$

$$= g(x,y,z) + (x'-x)\left.\frac{\partial g}{\partial x}\right|_x + (y'-y)\left.\frac{\partial g}{\partial y}\right|_y + (z'-z)\left.\frac{\partial g}{\partial z}\right|_z + \ldots$$

$$= g(\vec{r}) + \left[(\vec{r}\,'-\vec{r})\cdot\vec{\nabla}\right] g$$

differentiate w.r.t. $\vec{r}$ unprimed.

Extending this to vector functions gives:

$$\vec{E}_{ext}(\vec{r}\,') = \vec{E}_{ext}(\vec{r}) + \left[(\vec{r}\,'-\vec{r})\cdot\vec{\nabla}\right]\vec{E}_{ext}(\vec{r})$$

Here $\vec{r}$ can be an arbitrary location. So

$$\vec{F} = \int \rho(\vec{r}\,')\left\{\vec{E}_{ext}(\vec{r}) + \left[(\vec{r}\,'-\vec{r})\cdot\vec{\nabla}\right]\vec{E}_{ext}(\vec{r})\right\} d\tau'$$

$$= \left\{\int \rho(\vec{r}\,')\, d\tau'\right\}\vec{E}_{ext}(\vec{r}) + \left\{\int \rho(\vec{r}\,')\left[(\vec{r}\,'-\vec{r})\cdot\vec{\nabla}\right] d\tau'\right\}\vec{E}_{ext}(\vec{r})$$

With no net charge

$$\int \rho(\vec{r}\,')\, d\tau' = 0$$

Then

$$\vec{F} = \left\{\int \rho(\vec{r}\,')\,\vec{r}\,'\cdot\vec{\nabla}\, d\tau'\right\}\vec{E}_{ext}(\vec{r}) - \left\{\int \rho(\vec{r}\,')\,\vec{r}\cdot\vec{\nabla}\, d\tau'\right\}\vec{E}_{ext}(\vec{r})$$

$$= \left\{\underbrace{\left[\int \rho(\vec{r}\,')\,\vec{r}\,' d\tau'\right]}_{=\vec{p}}\cdot\vec{\nabla}\right\}\vec{E}_{ext}(\vec{r}) - \underbrace{\int \rho(\vec{r}\,')\, d\tau'}_{=0 \text{ no net charge}}\left\{\vec{r}\cdot\vec{\nabla}\right\}\vec{E}_{ext}(\vec{r})$$

$$\Rightarrow \quad \vec{F} = \left[\vec{p}\cdot\vec{\nabla}\right]\vec{E}_{ext}(\vec{r})$$

In order for this to be valid we need the remaining terms small and this requires $(\vec{r}\,'-\vec{r})$ be small. So $\vec{r}$ must be in the vicinity of the charge distrib.

Net torque: The force at location $\vec{r}'$ in the distribution is:

$$d\vec{F} = \rho(\vec{r}')d\tau' \, \vec{E}_{ext}(\vec{r}')$$

The torque is

$$d\vec{\tau} = \vec{r}' \times d\vec{F}$$

$$= \vec{r}' \times \vec{E}_{ext}(\vec{r}') \, \rho(\vec{r}')d\tau'$$

The net torque is

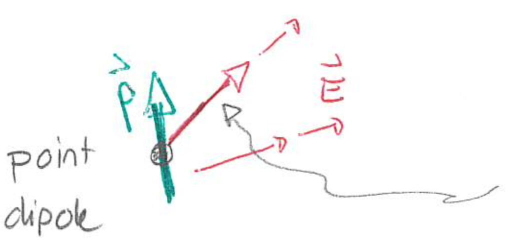
$$\tau = \int \rho(\vec{r}') \, \vec{r}' \times \vec{E}_{ext}(\vec{r}') \, d\tau'$$

For a uniform field $\vec{E}_{ext}(\vec{r}') = \vec{E}_{ext}$ and we get:

$$\vec{\tau} = \underbrace{\left\{ \int \rho(\vec{r}') \, \vec{r}' \, d\tau' \right\}}_{\vec{p}} \times \vec{E}_{ext}$$

$$\Rightarrow \tau = \vec{p} \times \vec{E}_{ext}. \qquad \blacksquare$$

Such results can be extended to the situation where the external field is not uniform. If the dipole is a point dipole then the integral extends over an infinitesimal volume and only the electric field at that point matters



only use this $\vec{E}$ vector at the dipole location.

An additional consideration is the potential energy associated with the point dipole in the field. In general if such a potential energy, $\mathcal{U}$, exists then the force on the dipole is:

$$\vec{F} = -\vec{\nabla}\mathcal{U}$$

We can use this to show:

$$\boxed{\begin{array}{l}\text{Suppose that a point dipole with dipole moment } \vec{p} \text{ is placed in an external electric field, } \vec{E}_{ext}\text{, Then the potential} \\ \text{energy associated with the interaction between the field and the dipole is} \\ \qquad \mathcal{U} = -\vec{p}\cdot\vec{E}_{ext}.\end{array}}$$

1 Dipole in an external electric field

A sphere with radius R contains charge whose distribution, in spherical coordinates, is

$$\rho(\mathbf{r}') = \frac{3\alpha}{4\pi R^3}\,\cos\theta'$$

where $\alpha > 0$ has units of Coulombs. The dipole moment for this has the form $\mathbf{p} = p\hat{\mathbf{z}}$ where p is a constant.

a) Suppose that the dipole is placed in the external electric field

$$\mathbf{E} = \beta z\,\hat{\mathbf{z}}$$

where $\beta > 0$ is a constant. Based on physical reasoning, what is the direction of the net force on the dipole? Calculate the net force on the dipole.

b) Suppose that the dipole is placed in the external electric field

$$\mathbf{E} = \beta \cancel{x\,\hat{\mathbf{z}}}\ (x+R)\,\hat{x}$$

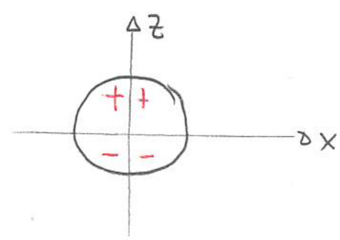
where $\beta > 0$ is a constant. Based on physical reasoning, what is the direction of the net force on the dipole? Calculate the net force on the dipole.

c) Suppose that the dipole is placed in the external electric field

$$\mathbf{E} = \beta \cancel{x\,\hat{\mathbf{z}}}\ (x+R)\,\hat{x}$$

where $\beta > 0$ is a constant and the charge distribution is allowed to reorient itself. How will it align so as to yield the smallest potential energy?

Answer: The distribution is

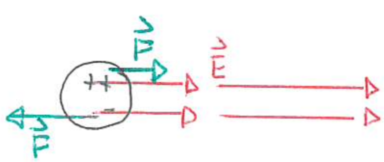


a)

Net force along $+\hat{z}$

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\,\vec{E} = p\frac{\partial}{\partial z}\vec{E} = p\beta\,\hat{z}$$ ✓

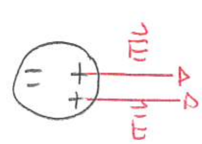
b)



Net force is zero

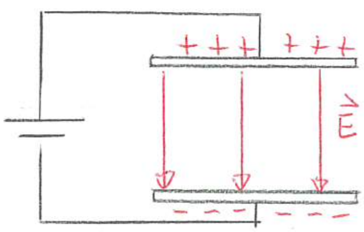
$$\vec{F} = p\frac{\partial}{\partial z}\vec{E} = 0$$

c) Need $U = -\vec{p}\cdot\vec{E}$ minimum $\Rightarrow$ $\vec{p}$ parallel to $\vec{E}$

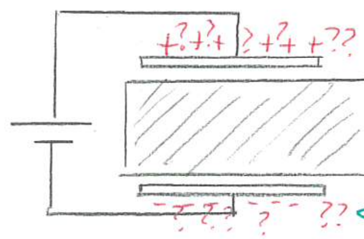


Dipoles in a material: polarization.

We now consider the effects of external fields on matter which is not a conductor but which contains a collection of dipoles. A simple example would be a sheet of material inserted between parallel capacitor plates



insert material



We expect that

1) the field due to the plates will reorient dipoles within the material.
2) the re-oriented dipoles will produce their own field, which will adjust the overall field.
3) the re-oriented dipoles will produce an adjusted charge density on the plates.

Demo: PhET Capacitor Lab

$\Rightarrow$ Dielectric Tab

* slide dielectric in out with battery connected

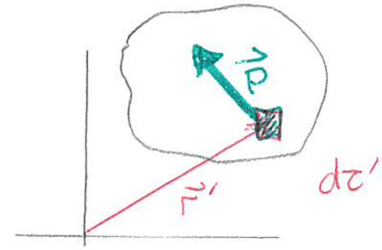
* disconnect battery - slide dielectric in/out

- observe charge
- observe polarization
- observe V

We will need an aggregate quantity to describe the effect of the dipoles. The idea will be

Polarization = dipole moment per unit volume (position dependent)

More precisely:

> The polarization of a material is a vector $\vec{P}(\vec{r}\,')$ such that the dipole moment in a small region at $\vec{r}\,'$ with volume $d\tau'$ is
>
> $$d\vec{p} = \vec{P}(\vec{r}\,')\,d\tau'$$
>
> This has units of C/m^2
>
> 
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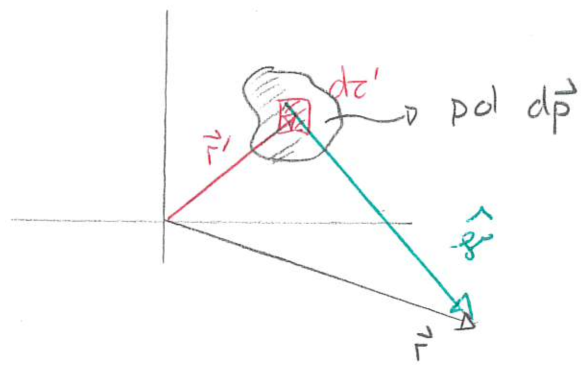
Potential from polarization

Electromagnetic theory already provides a mechanism for determining the potential produced by a point dipole and this can be extended to a material whose polarization is known. Specifically we break the distribution into segments. The shaded segment gives potential at location $\vec{r}$:



$$dV = \frac{1}{4\pi\epsilon_0}\frac{d\vec{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

Integration over all space gives:

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int_{\text{distribution}}\frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'}$$

We can prove:

> If a region has polarization $\vec{P}(\vec{r}')$ then the potential produced by the dipole distribution within the region is:
>
> $$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{surface}} \frac{\sigma_b(\vec{r}')}{\mathscr{r}} da' + \frac{1}{4\pi\epsilon_0} \int_{\text{region}} \frac{\rho_b(\vec{r}')}{\mathscr{r}} d\tau'$$
>
> where
>
> $$\rho_b(\vec{r}') = -\vec{\nabla}\cdot\vec{P}$$
>
> is the <u>bound volume charge density</u> and
>
> $$\sigma_b(\vec{r}') = \vec{P}\cdot\hat{n}$$
>
> is the bound surface charge density. Here $\hat{n}$ is the normal to the surface.

<u>Proof:</u> We use

$$\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

where differentiation is w.r.t. primed. So

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

Now

$$\vec{\nabla}'\cdot\left(\vec{P}(\vec{r}')\frac{1}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\vec{\nabla}'\cdot\vec{P}(\vec{r}') + \vec{P}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)$$

$$\Rightarrow \vec{P}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \vec{\nabla}'\cdot\left(\vec{P}(\vec{r}')\frac{1}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\vec{\nabla}'\cdot\vec{P}(\vec{r}')$$

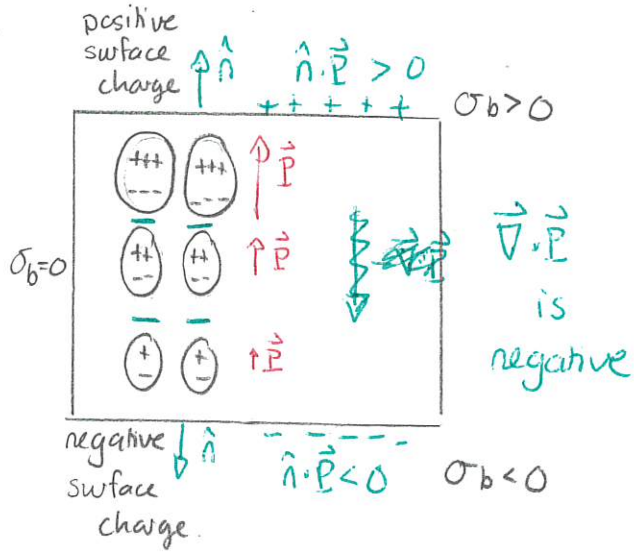
Thus

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{region}} \vec{\nabla}'\cdot\left(\vec{P}(\vec{r}\,')\frac{1}{r}\right) d\tau' - \frac{1}{4\pi\epsilon_0}\int_{\text{region}} \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')}{r}\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_{\text{surface}} \frac{\vec{P}(\vec{r}\,')\cdot\hat{n}}{r}\, d\tau' - \frac{1}{4\pi\epsilon_0}\int_{\text{region}} \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')}{r}\, d\tau'$$

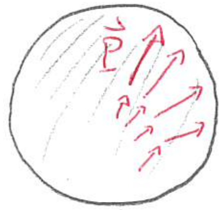
With the indicated definitions of bound volume and surface charge densities this gives the required result.

We can establish an intuitive idea for this by considering the illustrated distribution of dipoles. The gradient of dipoles clearly results in non-zero charge distributions within and on the surface. These are bound to the material and thus called <u>bound charges</u>.



So the scheme for determining the electrostatic potential and electric field is

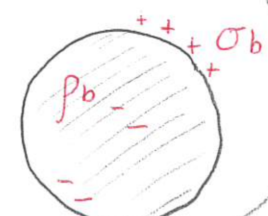
Describe polarization via $\vec{P}$



→ Determine bound surface + volume charge densities:

$$\rho_b = -\vec{\nabla}\cdot\vec{P}(\vec{r}\,')$$

$$\sigma_b = \hat{n}\cdot\vec{P}$$



→ Use usual methods for computing potential from charge densities ⇝ use ρ_b, σ_b

→ Use Coulomb's Law for getting $\vec{E}$ with bound surface + volume charge densities

→ Use Gauss' Law and bound charges to determine fields

2 Bound charge distributions: radial polarization

A sphere of radius R has polarization

$$\mathbf{P} = \alpha r^2 \hat{\mathbf{r}}$$

where $\alpha > 0$ is a constant with units $\mathrm{C/m^4}$.

a) Draw a qualitative sketch of the dipole distribution within the sphere.

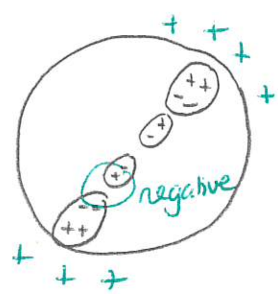
b) Do you expect that either the bound surface or bound volume charge densities will be zero? If not what do you expect their signs will be?

c) Determine the bound surface and volume charge densities.

d) Determine the total charge on the sphere.

e) Determine the electric field at all locations.

Answer: a)



b) It appears $\sigma_b > 0$

$\rho_b < 0$

c) Surface $\quad \sigma_b = \hat{n} \cdot \vec{P} = \hat{r}\, \alpha R^2 \hat{r} = \alpha R^2$

$$\boxed{\sigma_b = \alpha R^2} > 0$$

volume

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\left\{ \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 P_r\right) + \cdots \cancel{\frac{\partial}{\partial \theta}\left(\sin\theta P_\theta\right)} + \cdots \cancel{\frac{\partial P_\phi}{\partial \phi}} \right\}$$

$$= -\frac{1}{r^2} \frac{\partial}{\partial r}\left(\alpha r^4\right)$$

$$= -\frac{1}{r^2} \cdot 4\alpha r^3 \quad \Rightarrow \quad \boxed{\rho_b = -4\alpha r} < 0$$

d) On the surface

$$\int_{surface} \sigma_b \, da' = \int_{surface} \alpha R^2 \, da' = \alpha R^2 \int_{surface} da' = 4\pi\alpha R^2$$

Inside the sphere

$$\left.\begin{array}{l} 0 < r' \le R \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\} d\tau' = r'^2 \sin\theta' \, dr' d\theta' d\phi'$$

The charge inside is:

$$\int \rho_b \, d\tau' = \int_0^R dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \, r'^2 \sin\theta' \, (-4\alpha r')$$

$$= -4\alpha \underbrace{\int_0^R r'^3 dr'}_{\frac{R^4}{4}} \underbrace{\int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi'}_{4\pi} = -4\pi\alpha R^2$$

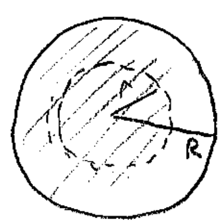
These sum to zero. So total charge is zero.

e) Use Gauss' Law. By symmetry

$$\vec{E} = E_r(r)\,\hat{r}$$

We use a Gaussian sphere with radius r.

On this sphere:



$$\left.\begin{array}{l} r = r \\ 0 \le \theta \le \pi \\ 0 \le \phi \le 2\pi \end{array}\right\} d\vec{a} = r^2 \sin\theta \, d\theta d\phi \, \hat{r}$$

$$\Rightarrow \quad \vec{E} \cdot d\vec{a} = r \sin\theta \, d\theta d\phi$$

Then:

$$\oint \vec{E}\cdot d\vec{a} = E_r(r)\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \; R^2 \sin\theta = 4\pi r^2 E_r(r)$$

$$\Rightarrow \quad E_r(r) = \frac{q_{enc}}{4\pi\epsilon_0 r^2}$$

Now if r is inside $(r<R)$, then

$$q_{enc} = \int \rho_b \, d\tau'$$

$$= -4\alpha \int_0^{r} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \; r'^2 \sin\theta' \; r' = -\alpha 4\pi \int_0^{r} r'^3 dr' 4$$

$$= -\frac{4\pi\alpha r^4 4}{4} = -\pi\alpha r^4 4$$

Then

$$E_r(r) = \frac{-4\pi\alpha r^4}{4\pi\epsilon_0 r^2} = -\frac{\alpha r^2}{\epsilon_0} \qquad r<R$$

If $r > R$ (outside) then $q_{enc} = 0$. Thus:

$$\vec{E}(\vec{r}) = \begin{cases} -\dfrac{\alpha r^2}{4\epsilon_0}\hat{r} & r<R \\ 0 & r>R \end{cases}$$