Statistical and Thermal Physics: Class Exam II

23 April 2020

Name: ______________________ **Total:** /50

Instructions

- There are 6 questions on 10 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$$R = 8.31\,\mathrm{J/mol\,K} \qquad N_A = 6.02\times10^{23}\,\mathrm{mol}^{-1} \qquad 1\,\mathrm{atm} = 1.01\times10^{5}\,\mathrm{Pa}$$

$$k = 1.38\times10^{-23}\,\mathrm{J/K} = 8.61\times10^{-5}\,\mathrm{eV/K} \qquad e = 1.60\times10^{-19}\,\mathrm{C}$$

$$N! \approx N^N e^{-N}\sqrt{2\pi N} \qquad \ln N! \approx N\ln N - N \qquad e^x \approx 1 + x \quad \text{if } x \ll 1$$

Question 1

A collection of nine identical spin-1/2 particles is in a region with zero magnetic field. Consider the following three macrostates of the system:

State	Number with spin up	Number with spin down	Ω
State A	3	6	$\binom{9}{3} = 84$
State B	6	3	$\binom{9}{6} = 84$
State C	4	5	$\binom{9}{4} = 126$

Rank these states in order of increasing likelihood. Indicate equality whenever this occurs. Explain your answer.

Likelihood proportional to multiplicity $\Omega = \binom{N}{n_+} = \binom{9}{n_+}$

$A = B < C$

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Question 2

2 a) An Einstein solid, labeled A, consists of 4 oscillators and 6 energy units. Determine the number of microstates of the system.

$$\Omega = \binom{N+q-1}{q} = \frac{(N+q-1)!}{(N-1)!\,q!} = \frac{(4+6-1)!}{(4-1)!\,6!} = \frac{9!}{3!\,6!} = 84$$

10 b) Another Einstein solid, labeled B, consists of 2 oscillators and 0 energy units. It is placed into contact with system A. The two systems can exchange energy. The macrostates of the combined system are described by specifying the number of energy units for A and the number for B. List all possible macrostates of the combined system and determine the probability with which each occurs.

$$q = q_A + q_B = 6 \equiv \text{fixed}$$

q_A	q_B	Ω_A	Ω_B	$\Omega = \Omega_A \Omega_B$	prob
0	6	1	7	7	7/462
1	5	4	6	24	24/462
2	4	10	5	50	50/462
3	3	20	4	60	60/462
4	2	35	3	105	105/462
5	1	56	2	112	112/462
6	0	84	1	84	84/462
				Total 462	

$$\Omega_A = \binom{4+q_A-1}{q_A} = \binom{3+q_A}{q_A} = \frac{(3+q_A)!}{q_A!\,3!}$$

$$\Omega_B = \binom{2+q_B-1}{q_B} = \binom{q_B+1}{q_B} = \frac{(q_B+1)!}{q_B!\,1!} = q_B+1$$

Question 2 continued ...

1 c) Describe the equilibrium macrostate after the systems have interacted.

Most likely $\Rightarrow$ $q_A = 5$ $q_B = 1$

although $q_A = 4$ $q_B = 2$ similar

2 d) Determine the change in the entropy of the systems from the moment that they start interacting until they reach equilibrium.

$S = k \ln \Omega$

Initially $\Omega = 84$

Later $\Omega = 112$

$$\Delta S = k \ln[112] - k \ln[84] = k \ln\left[\frac{112}{84}\right] = 0.29k$$

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Question 3

Answer either part a) or part b) for full credit for this problem.

a) A collection of N spin-1/2 particles, each with dipole moment μ is in a magnetic field with magnitude B. The temperature of the system is $T = 4\mu B/k$. Determine the energy of the system and describe whether there are more particles in the spin-up state than in the spin down state.

We need to use $\frac{1}{T} = \frac{\partial S}{\partial E}$ and $S = k \ln \Omega$. In this case we can express everything via $n_+ =$ number up. Then:

$\Omega = \binom{N}{n_+}$ and $\left.\begin{array}{l} -\mu B n_+ + \mu B n_- = E \\ n_+ + n_- = N \end{array}\right\} \Rightarrow E = \mu B(N - 2n_+) \Rightarrow \frac{E}{\mu B} = N - 2n_+ \Rightarrow 2n_+ = N - \frac{E}{\mu B}$, $n_+ = \frac{1}{2}\left(N - \frac{E}{\mu B}\right)$

so we will get n_+ in terms of temperature and use this to get E

Then

$$S = k \ln \Omega$$
$$= k\{\ln[N!] - \ln(N - n_+!) - \ln(n_+!)\}$$
$$\approx k\{N\ln N - \cancel{N} - (N-n_+)\ln(N-n_+) + \cancel{(N-n_+)} - n_+ \ln n_+ + \cancel{n_+}\}$$
$$= k\{N\ln N - (N-n_+)\ln(N-n_+) - n_+ \ln n_+\}$$

Then $\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{\partial S}{\partial n_+}\frac{\partial n_+}{\partial E}$

$$\frac{1}{T} = k\left\{\ln(N-n_+) - \frac{\cancel{N-n_+}}{\cancel{N-n_+}}(-1) - \ln n_+ - \frac{\cancel{n_+}}{\cancel{n_+}}\right\}\underbrace{\left(-\frac{1}{2\mu B}\right)}_{\frac{\partial n_+}{\partial E}}$$

$$\frac{1}{T} = -\frac{k}{2\mu B}\ln\left(\frac{N-n_+}{n_+}\right)$$

Here $T = \frac{4\mu B}{k} \Rightarrow -\frac{1}{2} = \ln\left(\frac{N-n_+}{n_+}\right)$

$$\Rightarrow e^{-1/2} = \frac{N-n_+}{n_+}$$
$$= \frac{N}{n_+} - 1$$
$$\Rightarrow \frac{N}{n_+} = 1 + e^{-1/2} = 1.6 \Rightarrow n_+ = \frac{N}{1+e^{-1/2}}$$
$$\Rightarrow \frac{n_+}{N} = 0.622$$

$\Rightarrow$ more in spin up

Now

$$E = \mu B\left(N - \frac{N}{1+e^{-1/2}}\right)$$
$$= N\mu B\left(\frac{e^{-1/2} - 1}{1+e^{-1/2}}\right)$$

Question 3 continued ...

$dE = TdS - PdV$

$TdS = dE + PdV$

$dS = \frac{1}{T}dE + \frac{P}{T}dU$

b) Consider a system of N particles that occupy volume V and for which the density of states is

$$g(E) = B\, V^{2N} E^{5N/2}$$

where E is the energy of the system and B is a constant that does not depend on E or V. Determine expressions for the temperature in terms E, V, N and also the pressure of the system in terms of T, V, N.

Need $\quad S = k \ln[g(E)\,dE] \quad$ and $\quad \frac{1}{T} = \frac{\partial S}{\partial E}$

$\frac{+P}{T} = \frac{\partial S}{\partial V}$

$$S = k \ln[B\, V^{2N} E^{5N/2} dE]$$

$$= k \ln[B] + 2Nk \ln V + \frac{5}{2} Nk \ln E + k \ln dE$$

so

$$\frac{\partial S}{\partial E} = \frac{5}{2} Nk \frac{1}{E} \Rightarrow \frac{1}{T} = \frac{5}{2}\frac{Nk}{E} \Rightarrow E = \frac{5}{2} NkT$$

Then $\quad \frac{P}{T} = \frac{\partial S}{\partial V} \Rightarrow \frac{P}{T} = \frac{2Nk}{V} \Rightarrow PV = 2NkT$

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Question 4

A physical system has two distinct states, labeled 1 and 2 and with energies E_1 and $E_2 > E_1$ respectively. The probability with which the system is in state 1 is p_1 and that with which it is in state 2 is p_2. Consider the ratio of the probabilities p_2/p_1. At some initial temperature it is found that $p_2/p_1 = 1/3$ Suppose that the temperature of the system is halved (i.e. $T \to T/2$). Which of the following is true?

i) $p_2/p_1 \to 1/9$
ii) $p_2/p_1 \to 1/6$
iii) $p_2/p_1 \to 1/3$
iv) $p_2/p_1 \to 3$
v) $p_2/p_1 \to 9$
vi) The ratio is unaltered.
vii) None of the above.

$$P_s = \frac{e^{-E_s \beta}}{Z}$$

so $$\frac{P_2}{P_1} = \frac{e^{-E_1\beta}}{e^{-E_2\beta}}$$

Explain your answer.

$$\Rightarrow \frac{P_2}{P_1} = e^{(E_2-E_1)/kT}$$

$$\left(\frac{P_2}{P_1}\right)_{initial} = e^{\frac{E_2-E_1}{k}\frac{1}{T_i}}$$

$$\left(\frac{P_2}{P_1}\right)_{final} = e^{\frac{E_2-E_1}{k}\frac{1}{T_f}} = e^{\frac{E_2-E_1}{k}\frac{2}{T_i}}$$

$$= \left(e^{\frac{E_2-E_1}{kT_i}}\right)^2$$

$$= \left(\frac{P_2}{P_1}\right)^2_{initial} = \frac{1}{9}$$

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Question 5

Answer either part a) or part b) for full credit for this problem.

a) Consider a system of N spin-1/2 particles, each in a magnetic field for which $\mu B = 1$ If n_+ denotes the number of particles in spin up then the energy of the collection is $E = N - 2n_+$. Show that if $N \gg n_+ \gg 1$ then the entropy is

$$S \approx kN \ln\left(\frac{N}{N - n_+}\right) + kn_+ \ln\left(\frac{N - n_+}{n_+}\right).$$

$$S = k \ln \Omega$$

$$\text{But } \Omega = \binom{N}{n_+} = \frac{N!}{n_+!\,(N-n_+)!}$$

$$S = k \ln\left[\frac{N!}{n_+!\,(N-n_+)!}\right] = k\left[\ln(N!) - \ln(n_+!) - \ln(N-n_+)!\right]$$

$$\approx k\left\{N\ln N - \cancel{N} - n_+ \ln n_+ + \cancel{n_+} - (N-n_+)\ln(N-n_+) + \cancel{N} - \cancel{n_+}\right\}$$

$$= k\left\{N \ln N - N\ln(N-n_+) - n_+ \ln n_+ + n_+ \ln(N-n_+)\right\}$$

$$= k\, N \ln\left(\frac{N}{N-n_+}\right) + kn_+ \ln\left(\frac{N-n_+}{n_+}\right)$$

Question 5 continued ...

b) For the canonical ensemble prove that

$$\overline{E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2}.$$

$$\overline{E^2} = \sum E_s^2 \, p_s$$

$$= \sum_s E_s^2 e^{-E_s\beta}/Z$$

$$= \frac{1}{Z} \sum_s E_s^2 e^{-E_s\beta}$$

$$E_s e^{-E_s\beta} = -\frac{\partial}{\partial\beta} e^{-E_s\beta}$$

$$E_s^2 e^{-E_s\beta} = -\frac{\partial}{\partial\beta} E_s e^{-E_s\beta}$$

$$= -\frac{\partial}{\partial\beta}\left(-\frac{\partial}{\partial\beta} e^{-E_s\beta}\right) = \frac{\partial^2}{\partial\beta^2} e^{-E_s\beta}.$$

$$\Rightarrow \overline{E}^2 = \frac{1}{Z} \frac{\partial^2}{\partial\beta^2} \underbrace{\sum_s e^{-E_s\beta}}_{Z}$$

$$= \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2}$$

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Question 6

A particle constitutes a system with two states having energies $-\epsilon$ and ϵ.

a) Determine an expression for the partition function for the system at temperature T.

$$Z = \sum e^{-E_s \beta}$$

$$= e^{-(-\epsilon)\beta} + e^{-\epsilon\beta}$$

$$= e^{\epsilon\beta} + e^{-\epsilon\beta}$$

$$= e^{\epsilon/kT} + e^{-\epsilon/kT}$$

Question 6 continued ...

b) Determine an expression for the mean value for the energy.

$$\bar{E} = -\frac{\partial}{\partial \beta} \ln Z = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{1}{Z}\epsilon \left(e^{\epsilon\beta} - e^{-\epsilon\beta}\right)$$

$$\bar{E} = -\epsilon \frac{e^{\epsilon\beta} - e^{-\epsilon\beta}}{e^{\epsilon\beta} + e^{-\epsilon\beta}}$$

c) Suppose that $\epsilon = 5 \times 10^{-21}$ J and $T = 300$ K. Consider a large system of such particles. Determine the fraction of particles in each of the two energy states.

Need $P_{+\epsilon} = \frac{e^{-\epsilon\beta}}{Z}$

$P_{-\epsilon} = \frac{e^{\epsilon\beta}}{Z}$

$$\epsilon/kT = \frac{5\times10^{-21}\,J}{1.38\times10^{-23}\,J/K \cdot 300K} = 1.21$$

$$Z = e^{\epsilon\beta} + e^{-\epsilon\beta} = e^{1.21} + e^{-1.21} = 3.65$$

$$P_{+\epsilon} = \frac{e^{1.21}}{3.65} = 0.92$$

$$P_{-\epsilon} = \frac{e^{-1.21}}{3.65} = 0.08$$

d) In most realistic situations, ϵ is much smaller. Suppose that $\epsilon \ll kT$. Determine an approximate expression (to lowest non-zero order in ϵ) for the mean value of the energy.

$\epsilon/kT \ll 1 \Rightarrow \epsilon\beta \ll 1 \Rightarrow e^{\epsilon\beta} \approx 1 + \epsilon\beta$

$e^{-\epsilon\beta} \approx 1 - \epsilon\beta$

$$\bar{E} = -\epsilon \frac{1+\epsilon\beta - (1-\epsilon\beta)}{2} = \frac{-\epsilon^2\beta}{2}$$

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