Statistical and Thermal Physics: Class Exam II

23 April 2019

Name: Solution **Total:** /50

Instructions

- There are 6 questions on 10 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$$R = 8.31\,\mathrm{J/mol\,K} \qquad N_A = 6.02 \times 10^{23}\,\mathrm{mol}^{-1} \qquad 1\,\mathrm{atm} = 1.01 \times 10^5\,\mathrm{Pa}$$

$$k = 1.38 \times 10^{-23}\,\mathrm{J/K} = 8.61 \times 10^{-5}\,\mathrm{eV/K} \qquad e = 1.60 \times 10^{-19}\,\mathrm{C}$$

$$N! \approx N^N e^{-N}\sqrt{2\pi N} \qquad \ln N! \approx N \ln N - N \qquad e^x \approx 1 + x \quad \text{if } x \ll 1$$

Question 1

An isolated Einstein solid consists of four oscillators, labeled A, B, C and D. Any microstate of the solid can be described by listing the *energy level* n for each. These are denoted $n_A, n_B, n_C, \ldots$. Consider the following microstates:

State	n_A	n_B	n_C	n_D
microstate 1	1	1	1	1
microstate 2	4	0	0	0
microstate 3	0	2	0	2

a) Provide a value for q for each microstate. 2

$q = n_A + n_B + n_C + n_D = 4$ for all three microstates

b) Rank the microstates in order of increasing probability, indicating equality whenever this occurs. 2

All have same energy $\Rightarrow$ all equally probable

/4

Question 2

A system consists of four non-interacting identical spin-1/2 particles. The bulk observables for the system are the total energy of the ensemble and the particle number. For any *individual particle* the following are the possible states (where $\epsilon > 0$):

State	Energy	Probability	
spin up	$-\epsilon$	$\frac{7}{10}$	$\rightarrow P_+$
spin down	ϵ	$\frac{3}{10}$	$\rightarrow P_-$

a) List all possible macrostates and the probabilities with which each occurs.

Use number of particles in spin up N_+. Then number in spin down is $4-N_+$. Energy is $-\varepsilon N_+ + \varepsilon N_- = \varepsilon(4-2N_+)$

+2

N_+	Energy	Prob $= \binom{N}{N_+} P_+^{N_+} P_-^{N_-}$
0	4ε	$\binom{4}{0}\left(\frac{7}{10}\right)^0\left(\frac{3}{10}\right)^4 = 81/10000$
1	2ε	$\binom{4}{1}\left(\frac{7}{10}\right)\left(\frac{3}{10}\right)^3 = 756/10000$
2	0	$\binom{4}{2}\left(\frac{7}{10}\right)^2\left(\frac{3}{10}\right)^2 = 2646/10000$
3	-2ε	$\binom{4}{3}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^1 = 4116/10000$
4	-4ε	$\binom{4}{4}\left(\frac{7}{10}\right)^4\left(\frac{3}{10}\right)^0 = 2401/10000$

b) Determine the energy of the equilibrium state of this system.

Equil = most probable $\Rightarrow$ $N_+ = 3$ ~~4~~ +2

energy -2ε

/10

Question 3

Answer either part a) or part b) for full credit for this problem.

a) Two Einstein solids, A and B, interact with each other but are otherwise isolated. In the high temperature limit the multiplicity of A is $\Omega_A = \kappa_A q_A^{N_A}$ where N_A is the number of particles in solid A, q_A is the number of energy units in solid A and κ_A is a term that is independent of q_A. Similarly for B, $\Omega_B = \kappa_B q_B^{N_B}$. Determine an expression for the entropy of the combined system (in terms of $N_A, N_B, q_A, q_B, \kappa_A$ and κ_B). Describe *how* you could use this to determine the ratio in which the energy is shared between the two solids in the equilibrium state.

$$S = k \ln[\Omega]$$

$$= k \ln[\Omega_A \Omega_B] = k \ln \Omega_A + k \ln \Omega_B$$

$$= k \ln(\kappa_A q_A^{N_A}) + k \ln(\kappa_B q_B^{N_B})$$

$$S = k\{\ln \kappa_A + \ln \kappa_B + N_A \ln q_A + N_B \ln q_B]$$

Determine maximum value for S. Note that if energy is fixed then $q = q_A + q_B$ is constant. So $q_B = q - q_A$ gives

$$S = k\left\{\ln \kappa_A + \ln \kappa_B + N_A \ln q_A + N_B \ln(q - q_A)\right\}$$

Differentiate w.r.t. q_A and set equal to zero

$$\Rightarrow \quad \frac{N_A}{q_A} - \frac{N_B}{q - q_A} = 0 \quad \Rightarrow \quad \frac{q_A}{N_A} = \frac{q_B}{N_B}$$

$$\Rightarrow \quad \frac{q_A}{q_B} = \frac{N_A}{N_B}$$ gives ratio

Question 3 continued . . .

b) Consider a single two dimensional quantum oscillator for which the states are labeled by integers n_x and n_y; these can independently take on the values $0, 1, 2, \ldots$. The energy for this is

$$E = \hbar\omega\ (n_x + n_y + 1)\,.$$

List the three macrostates with lowest energy and all the microstates for each. Describe *how* you would find the entropy, S, for an ensemble of N such identical, distinguishable, non-interacting, oscillators as a function of energy, E.

	n_x	n_y	E
macrostate 1	0	0	$\hbar\omega$
macrostate 2	0	1	$2\hbar\omega$
	1	0	$2\hbar\omega$
macrostate 3	1	1	$3\hbar\omega$
	0	2	$3\hbar\omega$
	2	0	$3\hbar\omega$

Count all states with energy E $\rightarrow$ multiplicity $\Omega(E,N)$

$\rightarrow$ $S = k \ln[\Omega(E,N)]$

/6

Question 4

Consider an Einstein solid, with N oscillators, for which $q \gg N \gg 1$ (this is the high temperature limit). Then the multiplicity of any macrostate is

$$\Omega \approx \left(\frac{eq}{N}\right)^N \frac{1}{\sqrt{2\pi N}}$$

and the energy of this state is $E = \hbar\omega\,(q + N/2)$.

a) Determine an expression for the entropy of the system.

$$S = k \ln \Omega$$

$$= k \ln \left\{ \left(\frac{eq}{N}\right)^N \frac{1}{\sqrt{2\pi N}} \right\}$$

$$= k \left\{ \ln \left[\left(\frac{eq}{N}\right)^N \right] + \ln \left[\frac{1}{\sqrt{2\pi N}} \right] \right\}$$

$$= k \left\{ N \ln \left(\frac{eq}{N}\right) - \frac{1}{2} \ln (2\pi N) \right\}$$

$$= k \left\{ N \left(\overset{1}{\cancel{\ln e}} + \ln q - \ln N\right) - \frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln N \right\}$$

$$= k \left\{ N \ln q - N \ln N + N - \underbrace{\frac{1}{2} \ln N - \frac{1}{2} \ln(2\pi)}_{\text{negligible}} \right\}$$

$$S \approx k \left(N \ln q - N \ln N + N \right)$$

Question 4 continued ...

b) Determine an expression for the energy of the solid, E, in terms of temperature and particle number.

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

$$= \frac{\partial S}{\partial q} \frac{\partial q}{\partial E} \qquad \frac{\partial q}{\partial E} = 1/\frac{\partial E}{\partial q} = \frac{1}{\hbar\omega}$$

$$= \frac{1}{\hbar\omega} Nk/q$$

$$\Rightarrow \quad q = \frac{NkT}{\hbar\omega}$$

$$\Rightarrow \quad E = \hbar\omega q + \hbar\omega N/2$$

$$\approx \hbar\omega q$$

$$\Rightarrow \quad E = NkT$$

/12

Question 5

An ensemble of N identical, distinguishable, non-interacting spin-1/2 particles are all in a magnetic field with magnitude B. The energy of a single particle in the spin-up state is $-\mu B$ and that of a single particle in the spin down state is $+\mu B$. These are in contact with a bath at temperature T.

a) Determine an expression for the partition function for the entire system.

$$Z_{single} = \sum e^{-E_s \beta}$$

$$Z = (Z_{single})^N$$

States

s	E_s
up	$-\mu B$
down	$+\mu B$

$$Z_{single} = e^{\beta \mu B} + e^{-\beta \mu B}$$

So $$Z = \left[e^{\beta \mu B} + e^{-\beta \mu B}\right]^N$$

Question 5 continued ...

b) Determine an expression for the mean value of the energy of the system.

$$\bar{E} = -\frac{\partial}{\partial \beta} \ln Z$$

$$= -\frac{\partial}{\partial \beta} \ln [\ldots]^N$$

$$= -\frac{\partial}{\partial \beta} N \ln \left[e^{B\mu\beta} + e^{-B\mu\beta} \right]$$

$$= -N \frac{B\mu \left[e^{B\mu\beta} - e^{-B\mu\beta} \right]}{e^{B\mu\beta} + e^{-B\mu\beta}}$$

$$= -NB\mu \frac{\sinh B\mu\beta}{\cosh B\mu\beta}$$

$$\bar{E} = -NB\mu \tanh[B\mu\beta]$$

/12

Question 6

Answer either part a) or part b) for full credit for this problem.

a) Consider an ensemble of identical, non-interacting particles at temperature T. Each can be in one of four distinct states; the energies of these states are $0, \epsilon, \epsilon, \epsilon$ where $\epsilon = kT/4$. Determine the probability with which any single particle will be in the lowest energy state.

$$\text{prob} = e^{-E_s\beta}/Z$$

$$= \frac{e^{-0\beta}}{Z} = \frac{1}{Z}$$

$$Z = \sum e^{-E_s\beta}$$

$$= e^{-0\beta} + e^{-\epsilon\beta} + e^{-\epsilon\beta} + e^{-\epsilon\beta}$$

$$= 1 + 3e^{-\epsilon\beta}$$

$$= 1 + 3e^{-kT/4kT} = 1 + 3e^{-0.25} = 3.336$$

$$\text{prob} = \frac{1}{3.336} = 0.30$$

Question 6 continued ...

b) Using the canonical ensemble, show that the heat capacity for a system is

$$c = \frac{1}{kT^2}\frac{\partial^2 \ln Z}{\partial \beta^2}$$

and use this to show that for a system consisting of N identical, distinguishable, non-interacting particles that the heat capacity is proportional to N.

$$C = \frac{\partial \bar{E}}{\partial T} = \frac{\partial \bar{E}}{\partial \beta}\frac{\partial \beta}{\partial T}$$

$$\beta = \frac{1}{kT}$$

$$\frac{\partial \beta}{\partial T} = -\frac{1}{kT^2}$$

$$= -\frac{1}{kT^2}\frac{\partial}{\partial \beta}\bar{E}$$

But $\quad \bar{E} = -\frac{\partial}{\partial \beta}\ln Z \quad \Rightarrow \quad C = \frac{1}{kT^2}\frac{\partial^2 \ln Z}{\partial \beta^2}$

Here $\quad Z = (Z_{single})^N \quad \Rightarrow \quad C = \frac{1}{kT}\frac{\partial^2}{\partial \beta^2}\ln(Z_{single})^N$

$$= N/kT \underbrace{\frac{\partial^2 \ln(Z_{single})}{\partial \beta^2}}_{\text{independent of } N}$$

$\Rightarrow$ proportional to N

/6