Statistical and Thermal Physics: Class Exam II

23 April 2019

Name: ____________________________ **Total:** /50

Instructions

- There are 6 questions on 10 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$$R = 8.31\,\text{J/mol K} \qquad N_A = 6.02 \times 10^{23}\,\text{mol}^{-1} \qquad 1\,\text{atm} = 1.01 \times 10^{5}\,\text{Pa}$$

$$k = 1.38 \times 10^{-23}\,\text{J/K} = 8.61 \times 10^{-5}\,\text{eV/K} \qquad e = 1.60 \times 10^{-19}\,\text{C}$$

$$N! \approx N^N e^{-N}\sqrt{2\pi N} \qquad \ln N! \approx N \ln N - N \qquad e^x \approx 1 + x \quad \text{if } x \ll 1$$

Question 1

An isolated Einstein solid consists of four oscillators, labeled A, B, C and D. Any microstate of the solid can be described by listing the *energy level* n for each. These are denoted $n_A, n_B, n_C, \ldots$. Consider the following microstates:

State	n_A	n_B	n_C	n_D
microstate 1	1	1	1	1
microstate 2	4	0	0	0
microstate 3	0	2	0	2

a) Provide a value for q for each microstate.

b) Rank the microstates in order of increasing probability, indicating equality whenever this occurs.

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Question 2

A system consists of four non-interacting identical spin-1/2 particles. The bulk observables for the system are the total energy of the ensemble and the particle number. For any *individual particle* the following are the possible states (where $\epsilon > 0$):

State	Energy	Probability
spin up	$-\epsilon$	$\frac{7}{10}$
spin down	ϵ	$\frac{3}{10}$

a) List all possible macrostates and the probabilities with which each occurs.

b) Determine the energy of the equilibrium state of this system.

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Question 3

Answer either part a) or part b) for full credit for this problem.

a) Two Einstein solids, A and B, interact with each other but are otherwise isolated. In the high temperature limit the multiplicity of A is $\Omega_A = \kappa_A q_A^{N_A}$ where N_A is the number of particles in solid A, q_A is the number of energy units in solid A and κ_A is a term that is independent of q_A. Similarly for B, $\Omega_B = \kappa_B q_B^{N_B}$. Determine an expression for the entropy of the combined system (in terms of $N_A, N_B, q_A, q_B, \kappa_A$ and κ_B). Describe *how* you could use this to determine the ratio in which the energy is shared between the two solids in the equilibrium state.

Question 3 continued . . .

b) Consider a single two dimensional quantum oscillator for which the states are labeled by integers n_x and n_y; these can independently take on the values $0, 1, 2, \ldots$. The energy for this is

$$E = \hbar\omega \, (n_x + n_y + 1) \, .$$

List the three macrostates with lowest energy and all the microstates for each. Describe *how* you would find the entropy, S, for an ensemble of N such identical, distinguishable, non-interacting, oscillators as a function of energy, E.

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Question 4

Consider an Einstein solid, with N oscillators, for which $q \gg N \gg 1$ (this is the high temperature limit). Then the multiplicity of any macrostate is

$$\Omega \approx \left(\frac{eq}{N}\right)^N \frac{1}{\sqrt{2\pi N}}$$

and the energy of this state is $E = \hbar\omega\,(q + N/2)$.

a) Determine an expression for the entropy of the system.

Question 4 continued . . .

b) Determine an expression for the energy of the solid, E, in terms of temperature and particle number.

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Question 5

An ensemble of N identical, distinguishable, non-interacting spin-1/2 particles are all in a magnetic field with magnitude B. The energy of a single particle in the spin-up state is $-\mu B$ and that of a single particle in the spin down state is $+\mu B$. These are in contact with a bath at temperature T.

a) Determine an expression for the partition function for the entire system.

Question 5 continued ...

b) Determine an expression for the mean value of the energy of the system.

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Question 6

Answer either part a) or part b) for full credit for this problem.

a) Consider an ensemble of identical, non-interacting particles at temperature T. Each can be in one of four distinct states; the energies of these states are $0, \epsilon, \epsilon, \epsilon$ where $\epsilon = kT/4$. Determine the probability with which any single particle will be in the lowest energy state.

Question 6 continued . . .

b) Using the canonical ensemble, show that the heat capacity for a system is

$$c = \frac{1}{kT^2} \frac{\partial^2 \ln Z}{\partial \beta^2}$$

and use this to show that for a system consisting of N identical, distinguishable, non-interacting particles that the heat capacity is proportional to N.

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