Statistical and Thermal Physics: Final Exam

14 May 2019

Name: Solution **Total:** /70

Instructions

- There are 8 questions on 12 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$$R = 8.31\,\mathrm{J/mol\ K} \qquad N_A = 6.02\times10^{23}\,\mathrm{mol}^{-1} \qquad 1\,\mathrm{atm} = 1.01\times10^{5}\,\mathrm{Pa}$$

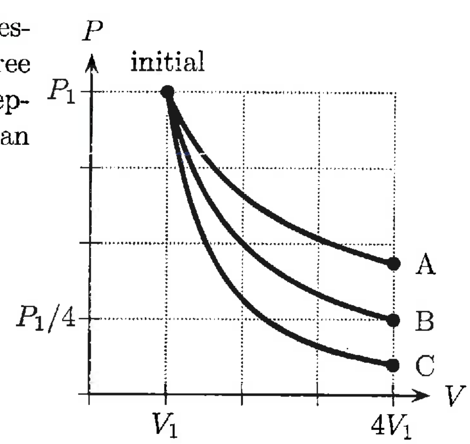
$$k = 1.38\times10^{-23}\,\mathrm{J/K} = 8.61\times10^{-5}\,\mathrm{eV/K} \qquad e = 1.60\times10^{-19}\,\mathrm{C} \qquad \hbar = 1.05\times10^{-34}\,\mathrm{Js}$$

$$N! \approx N^N e^{-N}\sqrt{2\pi N} \qquad \ln N! \approx N\ln N - N \qquad e^x \approx 1 + x \quad \text{if } x \ll 1$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}} \qquad \int_0^{\infty} x e^{-ax^2}\,\mathrm{d}x = \frac{1}{2a} \qquad \int_0^{\infty} x^2 e^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{4a^3}} \qquad \int_0^{\infty} x^3 e^{-ax^2}\,\mathrm{d}x = \frac{1}{2a^2}$$

Question 1

A monoatomic ideal gas initially has volume V_1 and pressure P_1. It can subsequently undergo any of the three indicated expansions. Describe which of these best represents an isothermal expansion and which represents an adiabatic expansion. Explain your choices.



In an adiabatic expansion temperature drops since

$$dE = \delta \cancel{Q}^{\,0} - PdV$$
$$= \underbrace{-PdV}_{\text{negative}} \quad \Rightarrow E \text{ drops}$$
$$\Rightarrow \tfrac{3}{2}NkT \text{ drops}$$

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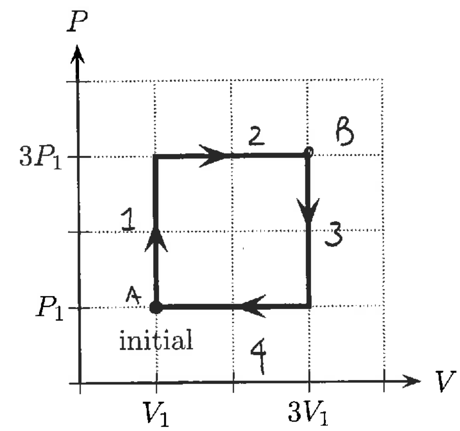
Along any isothermal $PV = NkT = \text{const}$

$\Rightarrow$ $\boxed{\text{B is isothermal}}$ +2

$\Rightarrow$ C has temp drop. $\Rightarrow$ adiabatic +3

Question 2

A heat engine constructed from a monoatomic ideal gas undergoes the cycle illustrated in the PV diagram.



a) Determine the efficiency of the engine in terms of P_i and V_i.

We need

$$\eta = \frac{W_{\text{by gas}}}{Q_{in}}$$ +1

We need work for each stage and Q for each stage. To do this note $\Delta E = Q + W$ where W is work done on the gas. Here

+1 $W = -\int P\,dV = -$ area under PV

+1 $E = \frac{3}{2}NkT = \frac{3}{2}PV$

Then stage by stage:

stage	ΔE	Q	W
1	$+3P_1V_1$	$3P_1V_1$	0
2	$+9P_1V_1$	$15P_1V_1$	$-6V_1P_1$
3	$-9P_1V_1$	$-9P_1V_1$	0
4	$-3P_1V_1$	$-5P_1V_1$	$2V_1P_1$
	+3	+2	+2

$Q = \Delta E - W$

Along stages 1,3 $dV = 0$ $\Rightarrow W = 0$

Along stage 2

$W = -$ area under PV

$= -[2V_1 \times 3P_1] = -6V_1P_1$

Along stage 3

$W = -$ area..

$= -[-2V_1 \times P_1] = 2V_1P_1$

Question 2 continued . . .

Then along stage 1

$$\Delta E = \frac{3}{2}\Delta(PV) = \frac{3}{2}(3P_1V_1 - P_1V_1) = 3P_1V_1$$

and stage 2

$$\Delta E = \frac{3}{2}\Delta(PV) = \frac{3}{2}(3V_1 3P_1 - 3P_1V_1) = 9P_1V_1$$

and stage 3

$$\Delta E = \frac{3}{2}\Delta(PV) = \frac{3}{2}(3V_1P_1 - 3P_1 3V_1) = -9P_1V_1$$

and stage 4

$$\Delta E = \frac{3}{2}\Delta(PV) = \frac{3}{2}(P_1V_1 - 3V_1P_1) = -3P_1V_1$$

The heat supplied arrives in stages 1,2 $\Rightarrow$ $Q_{in} = 18P_1V_1$ +2

The total work is $0 - 6V_1P_1 + 0 + 2V_1P_1 = -4V_1P_1$ $\Rightarrow$ $W_{by} = 4P_1V_1$ +1

$$\Rightarrow \eta = \frac{4P_1V_1}{18P_1V_1} = \frac{2}{9}$$ +1

b) Determine the optimal efficiency of the engine that operates between two heat reservoirs, such that the low temperature heat reservoir matches the lowest temperature attained by the gas during the cycle and the high temperature reservoir matches the highest temperature attained by the gas during the cycle.

+3

$$\eta \leq 1 - \frac{T_c}{T_h}$$

T_h is the temp at pt B $\qquad NkT_h = PV \Rightarrow NkT_h = 9P_1V_1$

T_c " " " " A $\qquad NkT_c = PV \Rightarrow NkT_c = P_1V_1$

$$\Rightarrow \frac{T_c}{T_h} = \frac{1}{9}$$

$$\Rightarrow \eta \leq 1 - \frac{1}{9} = 8/9$$

optimal is 8/9

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Question 3

Answer either part a) or part b) for full credit for this problem.

a) Show that, for any gas,
$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P.$$

$$dE = TdS - PdV$$

$$= T\left[\left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV\right] - PdV$$

$$= \underbrace{\left\{T\left(\frac{\partial S}{\partial V}\right)_T - P\right\}}_{\left(\frac{\partial E}{\partial V}\right)_T} dV + T\left(\frac{\partial S}{\partial T}\right)_V dT$$

b) A monoatomic ideal gas undergoes a free expansion in which its volume increases to 10 times the original volume. Determine the change in entropy of the gas.

$$S = \frac{3}{2}Nk\ln(T) + Nk\ln(V) + g(N)$$

← function of N only.

In a free expansion $T = \text{const}$

$$\Delta S = S_f - S_i = \cancel{\frac{3}{2}Nk\ln(T_f)} + Nk\ln(V_f)$$
$$- \cancel{\frac{3}{2}Nk\ln(T_i)} - Nk\ln(V_i)$$

$$= Nk\ln\left(\frac{V_f}{V_i}\right) = Nk\ln 10$$

$$\Delta S = Nk\ln 10$$

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Question 4

The entropy of a system of spin-1/2 particles, each with magnetic dipole moment μ and in magnetic field with magnitude B is

$$S = k\left\{N\ln(N) - \frac{1}{2}\left(N - \frac{E}{\mu B}\right)\ln\left[\frac{1}{2}\left(N - \frac{E}{\mu B}\right)\right] - \frac{1}{2}\left(N + \frac{E}{\mu B}\right)\ln\left[\frac{1}{2}\left(N + \frac{E}{\mu B}\right)\right]\right\}.$$

Determine an expression for the temperature of this system.

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{k}{2\mu B}\ln\left[\frac{1}{2}\left(N - \frac{E}{\mu B}\right)\right] \cancel{- \frac{k}{2}\left(N - \frac{E}{\mu B}\right)\frac{\frac{1}{2\mu B}}{\frac{1}{2}\left(N - E/\mu B\right)}}$$

$$- \frac{k}{2\mu B}\ln\left[\frac{1}{2}\left(N + \frac{E}{\mu B}\right)\right] \cancel{- \frac{k}{2}\left(N + \frac{E}{\mu B}\right)\frac{\frac{1}{2\mu B}}{\frac{1}{2}\left(N + E/\mu B\right)}}$$

$$\frac{1}{T} = \frac{k}{2\mu B}\ln\left[\frac{N - E/\mu B}{N + E/\mu B}\right]$$

$$\Rightarrow T = \frac{2\mu B}{k\ln\left[\frac{N - E/\mu B}{N + E/\mu B}\right]}$$

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Question 5

Consider two Einstein solids, labeled A and B, which contain identical oscillators. System A has four oscillators and system B has two oscillators. Initially each system is isolated and contains three energy units. The systems are then allowed to interact.

a) List the macrostates available to the interacting pair of Einstein solids. Provide the ~~probability with which~~ multiplicity for each macrostate ~~occurs~~.

There is a total of 6 energy units. Let q_A be the energy units for A, q_B energy units for B.

Then $q_A = 0, 1, 2, \ldots, 6$ and $q_B = 6 - q_A$.

The multiplicity is $\Omega_1(N,q) = \binom{N+q-1}{q}$

q_A	q_B	Ω_A	Ω_B	$\Omega = \Omega_A \Omega_B$
0	6	1	7	7
1	5	4	6	24
2	4	10	5	50
3	3	20	4	80
4	2	35	3	105
5	1	56	2	112
6	0	84	1	84

$\binom{q_A+3}{q_A}$ $\binom{q_B+1}{q_B} = q_B + 1$

Question 5 continued ...

b) Determine the equilibrium macrostate. Explain your answer.

One with largest multiplicity

$q_A = 5 \qquad q_B = 1$

although

$q_A = 4 \qquad q_B = 2$

is nearly as likely

c) In which direction does the energy flow as the system approaches equilibrium? Explain your answer.

From B to A $\rightarrow$ in equilibrium there are more energy units for A.

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Question 6

Answer either part a) or part b) for full credit for this problem.

a) An ensemble of 8 distingushable spin-1/2 particles is in a region in which there is no magnetic field. Let p_4 be the probability that the system is in the *macrostate* where exactly 4 particles have spin up and p_1 is the *macrostate* where exactly 1 particle has spin up. Someone claims that $p_4 = 4p_1$. Is this claim true or false? Explain your answer.

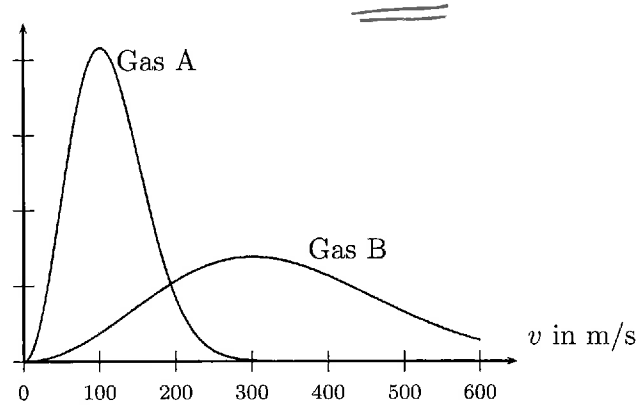
$$P_4 = \frac{1}{2^8}\binom{8}{4} = \frac{1}{2^8}\frac{8!}{4!4!} = \frac{70}{2^8}$$

$$P_1 = \frac{1}{2^8}\binom{8}{1} = \frac{1}{2^8}8 = \frac{8}{2^8}$$

$$\frac{P_4}{P_1} = \frac{70}{8} = \frac{35}{4} \neq 4$$

False

b) The distributions of speed for two gases, whose particles are the same, are provided. Describe as accurately as possible how the temperature of gas A is related to the temperature of gas B. Explain your answer.



most probable speed

$$V_{most\ prob} = \sqrt{\frac{2kT}{m}}$$

$$V_{most\ prob}^2 = \frac{2kT}{m}$$

$$\Rightarrow \frac{T_B}{T_A} = \left(\frac{V_{most\ prob\ B}}{V_{most\ prob\ A}}\right)^2 = 3^2 = 9$$

$\Rightarrow T_B = 9T_A$

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Question 7

Consider an ensemble of N distinguishable harmonic oscillators, each with the same energy,

$$E = \hbar\omega\left(n + \frac{1}{2}\right).$$

The ensemble is at equilibrium with a bath at temperature T.

a) Use the canonical ensemble formalism to show that the mean energy of the system is

$$\overline{E} = N\hbar\omega\left(\frac{1}{e^{\hbar\omega\beta} - 1} + \frac{1}{2}\right)$$

where $\beta = 1/kT$.

+1 $\left[\ \overline{E} = -\frac{\partial}{\partial\beta}\ln(Z_N) \quad \text{and} \quad Z_N = Z_{single}^{\ N}\ \right]$ +1

$$\Rightarrow \quad \overline{E} = -\frac{\partial}{\partial\beta}\ln\left(Z_{single}^{\ N}\right) = \underbrace{-N\frac{\partial}{\partial\beta}\ln\left[Z_{single}\right]}_{+1}$$

Now

$$Z_{single} = \sum_{\substack{\text{single}\\ \text{particle states}}} e^{-E_s\beta} \quad +1$$

$$= \sum_{n=0}^{\infty} e^{-E_n\beta} = \sum_{n=0}^{\infty} e^{-\hbar\omega(n+1/2)\beta}$$

$$= e^{-\hbar\omega\beta/2}\sum_{n=0}^{\infty} e^{-\hbar\omega\beta n}$$

$$= e^{-\hbar\omega\beta/2}\frac{1}{1-e^{-\hbar\omega\beta}} \quad +2$$

Question 7 continued ...

Then $\ln Z_{single} = -\frac{\hbar\omega\beta}{2} - \ln\left(1-e^{-\hbar\omega\beta}\right)$

+2

$$\Rightarrow \quad \bar{E} = -N\left[-\frac{\hbar\omega}{2} - \frac{e^{-\hbar\omega\beta}\hbar\omega}{1-e^{-\hbar\omega\beta}}\right]$$

$$= N\hbar\omega\left[\frac{1}{e^{\hbar\omega\beta}-1} + \frac{1}{2}\right]$$

b) Determine an expression for the mean energy in the high temperature approximation, $kT \gg \hbar\omega$ and determine the heat capacity of the system in this limit.

$$e^{\hbar\omega\beta} \ll 1 \Rightarrow \underbrace{e^{\hbar\omega\beta} \approx 1+\hbar\omega\beta}_{+1}$$

+2

$$\bar{E} \approx N\hbar\omega\left[\frac{1}{1+\hbar\omega\beta-1} + \frac{1}{2}\right]$$

$$= N\hbar\omega\left[\frac{1}{\hbar\omega\beta} + \frac{1}{2}\right] \approx N\hbar\omega\frac{1}{\hbar\omega\beta} = NkT$$

+1

$$C = \frac{d\bar{E}}{dT} = Nk.$$

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Question 8

Answer either part a) or part b) for full credit for this problem.

a) Carbon monoxide, a diatomic molecule vibrates as a harmonic oscillator with angular frequency 4.04×10^{14} Hz. For the quantum harmonic oscillator, the energy levels are $\hbar\omega(n + 1/2)$ where $n = 0, 1, 2, \ldots$. Let $P(n)$ be the probability with which the molecule is in state n.

i) Show that the ratio of particles in the first excited state to those in the ground state is $P(1)/P(0)$ and use this to determine an expression for this ratio in terms of frequency, temperature and constants.

$$P_n = \frac{e^{-E_n\beta}}{Z}$$

$$P_1/P_0 = \frac{e^{-E_1\beta}}{e^{-E_0\beta}} = e^{-(E_1-E_0)\beta}$$

$$\Rightarrow \quad P_1/P_0 = e^{-\hbar\omega\beta}$$

ii) Determine this ratio if the temperature is 600 K and describe what will happen to this ratio as the temperature increases.

$$\hbar\omega\beta = \frac{\hbar\omega}{kT} = \frac{1.05 \times 10^{-34}\,\mathrm{Js} \times 4.04 \times 10^{14}\,\mathrm{Hz}}{1.38 \times 10^{-23}\,\mathrm{J/K} \times 600\,\mathrm{K}} = 5.12$$

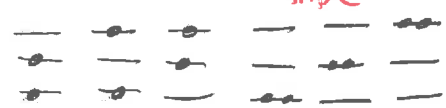
$$P_1/P_0 = e^{-\hbar\omega\beta} = 0.0060$$

As T increases β decreases $\Rightarrow$ ratio ~~approaches 1~~ increases.

Question 8 continued ...

b) i) A "toy" system has three states with energies ϵ_1, ϵ_2 and ϵ_3. Suppose that the system contains two particles. The particles could either be Bosons or Fermions. List the possible states available to the ~~pair~~ triple of particles for each case.

Boson



Fermions

ii) Consider a system of either Bosons or Fermions. Describe how the mean occupancy number of any state of the Boson system compares (e.g always equal, always larger, always smaller, sometimes larger, ect, ...) to that of the Fermion system. Explain your answer.

$$\bar{n}_B = \frac{1}{e^{(\epsilon-\mu)\beta} - 1} \qquad \text{Bosons}$$

$$\bar{n}_F = \frac{1}{e^{(\epsilon-\mu)\beta} + 1} \qquad \text{Fermions}$$

Since denom is smaller for Bosons $\bar{n}_B > \bar{n}_F$ always

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