Final: Monday 10 am - noon (12:30)

Format: * Covers Entire Semester.

* Similar to Previous Finals

* Print out, complete + turn in via D2L Drop Box.

Resources: * Notes, text

* No consulting any other person.

Review: Final 2017 all Q

Final 2019 all Q

1 Displacement current versus current

There are two possible sources for magnetic fields: current densities, $\mathbf{J}$ and time-varying electric fields. The time varying electric field contributes via

$$\mathbf{J}_d := \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

which is the displacement current density. This generates a displacement current via the usual surface integral.

One situation where the relative contributions of currents and displacement currents can be illustrated is a coaxial cable. This consists of a straight wire surrounded by a co-axial cylindrical shell with radius R. Suppose that current $I(t)$ flows down the wire and the same current returns in the reverse direction down the outer cylinder.

We will consider the fields between the two.

a) Using Ampère's law, determine the magnetic field produced by the current at all locations.

b) Now consider the induced electric field between the wire and cylindrical shell. Using

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

determine a set of differential equations for $\mathbf{E}$. Suggest a possible direction for $\mathbf{E}$, use this to simplify the equations and solve these for the electric field.

c) Determine an expression for the displacement current density and integrate this to obtain and expression for the total displacement current, I_d that flows between the wire and cylinder.

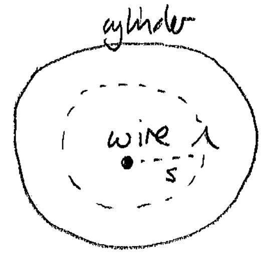
d) In the case where $I(t) = I_0 \cos(\omega t)$ determine the ratio I_d/I or a situation where the radius of the cylinder is 1 mm and the frequency 60 Hz (AC current). What does this imply about the effects of the displacement current?

e) Check whether your expressions for the electric and magnetic fields satisfy Maxwell's equations in the region between the wire and the cylinder. Under what circumstances are these correct?

Answer: a) $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

Assuming infinitely long wires the usual symmetry rules give

$\vec{B} = B_\phi(s)\,\hat{\phi}$



Using a circular path

$d\vec{l} = s\,d\phi'\hat{\phi} \qquad 0 \le \phi' \le 2\pi$

$$\oint \vec{B}\cdot d\vec{l} = 2\pi s\, B_\phi(s)$$

$$\Rightarrow \quad B_\phi(s) = \frac{\mu_0}{2\pi s} I_{enc}$$

So

$$\boxed{\vec{B} = \begin{cases} \dfrac{\mu_0}{2\pi s} I(t)\,\hat{\phi} & 0 < s < R \\ 0 & R < s \end{cases}}$$

b) $$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

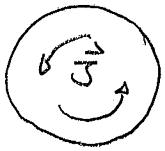
For $0 \le s \le R$

$$\vec{\nabla}\times\vec{E} = -\frac{\mu_0}{2\pi s}\frac{dI}{dt}\,\hat{\phi}$$

This is similar to $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ with $\vec{J} \to -\frac{1}{2\pi s}\frac{dI}{dt}\hat{\phi}$

So it's similar to producing a magnetic field from a current

This is akin to a solenoid. In that case $\vec{B}$ is along the z-axis. So here



$$\vec{E} = E_z(s)\,\hat{z}$$

Then

$$\vec{\nabla} \times \vec{E} = -\frac{\partial E_z}{\partial s}\,\hat{\phi} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial E_z}{\partial s} = \frac{\mu_0}{2\pi s}\frac{dI}{dt}$$

Then integrating gives:

$$E_z = \begin{cases} \dfrac{\mu_0}{2\pi}\dfrac{dI}{dt}\ln(s) + \text{constant} & \text{inside} \\ 0 & \text{outside.} \end{cases}$$

We want $\vec{E}$ to be continuous at $s=R$

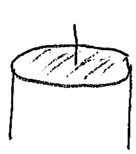
$$\Rightarrow \quad E_z(R) = 0 \quad \Rightarrow \quad \frac{\mu_0}{2\pi}\frac{dI}{dt}\ln R + \text{const} = 0$$

$$\Rightarrow \quad \text{const} = \frac{\mu_0}{2\pi}\frac{dI}{dt}\ln(R)$$

$$\Rightarrow \quad \boxed{\vec{E} = \begin{cases} \dfrac{\mu_0}{2\pi}\dfrac{dI}{dt}\ln\left[\dfrac{s}{R}\right]\hat{z} & 0 < s \leq R \\ 0 & R < s \end{cases}}$$

c) $$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\epsilon_0 \mu_0}{2\pi}\frac{d^2 I}{dt^2}\ln\left[\frac{s}{R}\right]\hat{z}$$

Then across the shaded surface $I_d = \int \vec{J}_d \cdot d\vec{a}$



$$d\vec{a} = s\,ds\,d\phi\,\hat{z}$$

So $\quad \vec{J}_d \cdot d\vec{a} = \dfrac{\epsilon_0 \mu_0}{2\pi} \dfrac{d^2 I}{dt^2} \, s \ln\left[\dfrac{s}{R}\right] ds\, d\phi$

$$\Rightarrow \quad I_d = \int_0^R \frac{\epsilon_0 \mu_0}{2\pi} \frac{d^2 I}{dt^2} \, s \ln\left[\frac{s}{R}\right] ds \quad \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \epsilon_0 \mu_0 \frac{d^2 I}{dt^2} \int_0^R s \ln\left[\frac{s}{R}\right] ds$$

$$\int s \ln\left(\frac{s}{R}\right) ds = \underbrace{\cancel{\frac{s^2}{2} \ln\left[\frac{s}{R}\right]\Big|_0^R}}_{0} - \int_0^R \frac{s^2}{2}\left[\frac{1}{s}\right] ds$$

$\int u\,dv = uv - \int v\,du$

$u = \ln \frac{s}{R}$

$dv = s$

$$= -\frac{1}{2} \int_0^R s\, ds = -\frac{R^2}{4}$$

Thus

$$I_d = -\epsilon_0 \mu_0 \frac{d^2 I}{dt^2} \frac{R^2}{4}$$

$$I_d = -\frac{\epsilon_0 \mu_0 R^2}{4} \frac{d^2 I}{dt^2}$$

d) $\quad \dfrac{d^2 I}{dt^2} = -\omega^2 I$

$$\Rightarrow \quad I_d = \frac{\epsilon_0 \mu_0 \omega^2 R^2}{4} I$$

$$\Rightarrow \quad \frac{I_d}{I} = \frac{\epsilon_0 \mu_0 \omega^2 R^2}{4}$$

Then $\epsilon_0 \mu_0 = 1/c^2$ gives:

$$\frac{I_d}{I} = \frac{1}{c^2} \frac{\omega^2 R^2}{4} = \left(\frac{\omega R}{2c}\right)^2$$

$$= \left(\frac{\cancel{2}\pi f R}{\cancel{2} c}\right)^2 = \left[\frac{\pi \times 60\,\text{Hz} \times 0.001\,\text{m}}{3 \times 10^8\,\text{m/s}}\right]^2$$

$$= 4.0 \times 10^{-19}$$

In terms of any observations the displacement current is miniscule.

e) Here for $s < R$

$$\vec{E} = \frac{\mu_0}{2\pi} \frac{dI}{dt} \ln\left[\frac{s}{R}\right] \hat{z}$$

$$\vec{B} = \frac{\mu_0}{2\pi s} I(t)\, \hat{\phi}$$

for $s > R$

$$\vec{E} = 0$$

$$\vec{B} = 0$$

Maxwells equations automatically satisfied if $\rho = 0$, $\vec{J} = 0$ ✓

We can check

$\vec{\nabla} \cdot \vec{E} = 0$ ✓

$$\vec{\nabla} \times \vec{E} = \frac{\mu_0}{2\pi} \dot{I} \left(-\frac{\partial E_z}{\partial s}\right) \hat{\phi}$$

$$= -\frac{\mu_0}{2\pi s} \dot{I}\, \hat{\phi} = -\frac{\partial \vec{B}}{\partial t}$$ ✓

$\vec{\nabla} \cdot \vec{B} = 0$ ✓

$$\vec{\nabla} \times \vec{B} = \frac{1}{s}\left[\frac{\partial}{\partial s}(s B_\phi)\right] = 0$$

$$\stackrel{?}{=} \cancel{\mu_0 \vec{J}} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \mu_0 \frac{\mu_0}{2\pi} \frac{d^2 I}{dt^2} \ln\left[\frac{s}{R}\right] \hat{z}$$

→ only works if $\frac{d^2 I}{dt^2} = 0$