Phys 311 2020

Lecture 41

Thurs HW by 5pm

Fri ZOOM

Consistency of the field equations

So far we have as field equations:

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

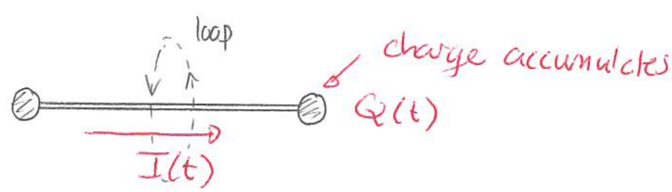
$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

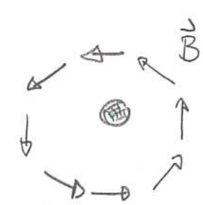
with the current and charge densities required to satisfy the continuity equation:

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

We will see that these are not consistent both mathematically and physically. First consider a physical situation of a current along a finite straight segment

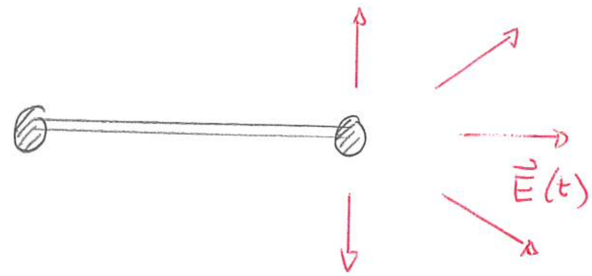


side view



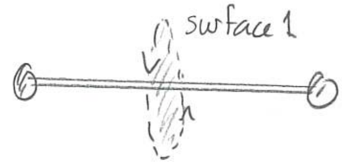
right end view

The Biot-Savart law predicts that the current will produce a magnetic field that circles the wire. Separately Coulomb's law predicts that the point charges at the ends produce a time-varying electric field, which will have a structure similar to an electric dipole field

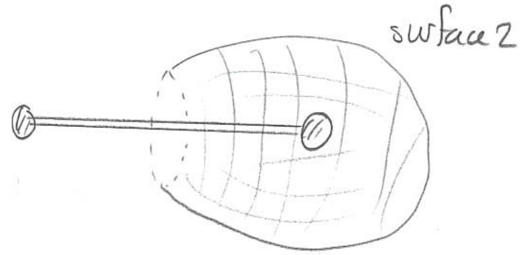


Now consider a line integral along a circular loop in a plane perpendicular to the wire. Mathematically

$$\oint_{\text{loop}} \vec{B}\cdot d\vec{l} = \int_{\text{surface}} \vec{\nabla}\times\vec{B}\cdot d\vec{a}$$



$$= \mu_0 \int_{\text{surface}} \vec{J}\cdot da$$



where the surface is any surface with the loop as its boundary. Thus we get

$$\int_{\text{surface 1}} \vec{J}\cdot d\vec{a} \neq 0 \qquad \text{and} \qquad \int_{\text{surface 2}} \vec{J}\cdot d\vec{a} = 0$$

Depending on which surface we use we get

$$\oint \vec{B}\cdot d\vec{l} \neq 0 \qquad \text{(surface 1)}$$

$$\oint \vec{B}\cdot d\vec{l} = 0 \qquad \text{(surface 2)}$$

But the magnetic fields cannot depend on the surface and the rule

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

cannot be correct in this case.

When could this rule be correct? An alternative piece of mathematics is that, for any vector field $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$. Then we check this for electric and magnetic fields:

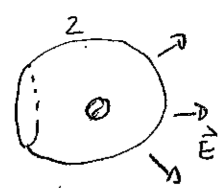
Electric	Magnetic
$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right)$	$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J})$
$= -\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0}$	$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$
$= 0$	$\Rightarrow \vec{\nabla}\cdot\vec{J}=0 \quad \Rightarrow \frac{\partial\rho}{\partial t}=0$

Thus the field equations so far are only correct when $\vec{\nabla}\cdot\vec{J}=0$ and $\frac{\partial\rho}{\partial t}=0$. This is the electrostatic case. So

$$\boxed{\begin{array}{lll} \vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0 & \vec{\nabla}\cdot\vec{B}=0 & \\ \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B}=\mu_0\vec{J} & \text{are only true if } \frac{\partial\rho}{\partial t}=0 \text{ and } \vec{\nabla}\cdot\vec{J}=0 \end{array}}$$

These are clearly not the case for the finite wire example. How could they be modified? We note that if $\frac{\partial\vec{E}}{\partial t}$ contributed as a source to the magnetic field then this would add to the integrals

$$\int \vec{\nabla}\times\vec{B}\cdot d\vec{a}$$



and thus we may get non-zero for the integral over the second surface.

Magnetic field induction

Suppose that there is an additional source of magnetic field $\vec{Z}$.
Then:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \vec{Z}$$

Now

$$\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}_{0} = \mu_0 \underbrace{\vec{\nabla} \cdot \vec{J}}_{-\frac{\partial \rho}{\partial t}} + \vec{\nabla} \cdot \vec{Z}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{Z} = \mu_0 \frac{\partial \rho}{\partial t}$$

$$= \mu_0 \frac{\partial}{\partial t} \epsilon_0 \vec{\nabla} \cdot \vec{E}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{Z} = \vec{\nabla} \cdot \left[\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

So one possible solution is $\vec{Z} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. Then:

$$\boxed{\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned}}$$

are the equations for magnetic fields. We see that a time-varying electric field can induce a magnetic field. In the case where $\vec{J} = 0$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

which means fields can be determined in the usual way by replacing
$\vec{J} \to \epsilon_0 \frac{\partial \vec{E}}{\partial t}$.

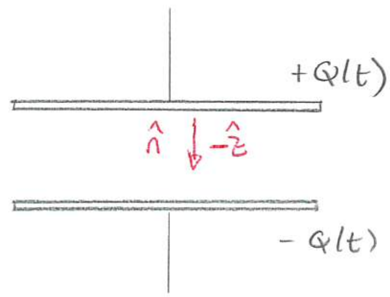
1 Fields between two capacitor plates

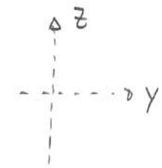
A parallel plate capacitor has circular plates with radius R and are separated by a small enough distance to regard as infinite. The plates are connected to a source that charges them with opposite charges and the charge on one plate is $Q(t)$. Assume that this is uniformly distributed on the plate and varies relatively slowly.

a) Determine an expression for the electric field produced between the plates by the charges on the plates.

b) Determine the magnetic field produced by the time-varying electric field at any location on a plane midway between the plates.

Answer: a)





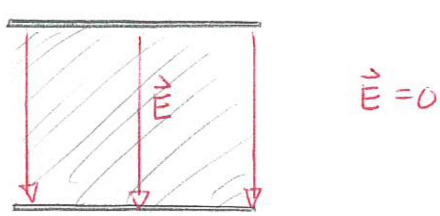
Between a pair of parallel plates,

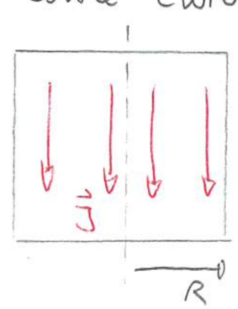
$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n} = \frac{Q(t)/A}{\epsilon_0}(-\hat{z}) \qquad A = \pi R^2$$

$$\Rightarrow \vec{E} = -\frac{Q(t)}{\epsilon_0 \pi R^2}\hat{z}$$

b) $$\vec{\nabla}\times\vec{B} = \underbrace{\mu_0 \cancel{\vec{J}}}_{\substack{0 \\ \text{between plates}}} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

Then, inside the plates $\vec{E}$ is uniform. This is analogous to a source current density





We will use an analog of Ampère's law to evaluate $\vec{B}$. First

$$\vec{B} = B_s \hat{s} + B_\phi \hat{\phi} + B_z \hat{z}$$

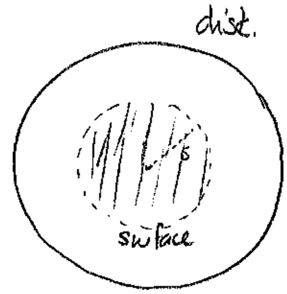
By the Biot-Savart type argument, $B_z = 0$. By a symmetry argument $B_s = 0$.
Thus

$$\vec{B} = B_\phi(s)\,\hat{\phi} \qquad \text{(only along midplane!)}$$

We can integrate along a circular loop of radius s. Then

$$\oint_{\text{loop}} \vec{B}\cdot\vec{dl} = \int_{\text{surface}} \vec{\nabla}\times\vec{B}\cdot d\vec{a}$$

$$= \mu_0\epsilon_0 \int_{\text{surface}} \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$$



For the l.h.s.

$$\left.\begin{array}{c} s' = s \\ 0 \le \phi' \le 2\pi \\ z' = 0 \end{array}\right\} \vec{dl} = s'd\phi'\,\hat{\phi} = s\,d\phi'\,\hat{\phi}$$

$$\Rightarrow \oint_{\text{loop}} \vec{B}\cdot\vec{dl} = \int_0^{2\pi} B_\phi(s)\, s\, d\phi' = 2\pi s\, B_\phi(s)$$

Thus

$$2\pi s\, B_\phi(s) = \mu_0\epsilon_0 \int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$$

$$\Rightarrow B_\phi(s) = \frac{\mu_0\epsilon_0}{2\pi s} \int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$$

Case 1 $\quad s \leq R$. The l.h.s requires

$$\left.\begin{array}{c} 0 \leq s' \leq s \\ 0 \leq \phi' \leq 2\pi \\ z' = 0 \end{array}\right\} \quad d\vec{a} = s' ds' d\phi' \hat{z}$$

$$B_\phi(s) = -\frac{\mu_0 \cancel{\epsilon_0}}{2\pi s} \int_0^s ds' \int_0^{2\pi} d\phi' s' \frac{dQ}{dt} \frac{1}{\cancel{\epsilon_0}\pi R^2}$$

$$= \frac{-\mu_0}{2\pi s} \frac{dQ}{dt} \frac{s^2}{R^2} \quad \Rightarrow \quad B_\phi(s) = \frac{-\mu_0 s}{2\pi R^2} \frac{dQ}{dt}.$$

Case 2 $\quad s \geq R$ The only difference is $0 \leq s' \leq R$. This gives

$$B_\phi(s) = \frac{-\mu_0}{2\pi s} \frac{dQ}{dt}$$

Thus we get

$$\vec{B} = \begin{cases} \dfrac{-\mu_0 s}{2\pi R^2} \dfrac{dQ}{dt} \hat{\phi} & s \leq R \\[2ex] \dfrac{-\mu_0}{2\pi s} \dfrac{dQ}{dt} \hat{\phi} & s \geq R \end{cases}$$

We could attempt to check $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ in the region between the plates. We know $B_z = 0$ and also none of the fields depends on ϕ. Then:

$$\vec{\nabla} \cdot \vec{B} = \frac{1}{s}\frac{\partial}{\partial s}(s B_s) + \cancel{\frac{1}{s}\frac{\partial B_\phi}{\partial \phi}} + \cancel{\frac{\partial B_z}{\partial z}}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}(s B_s)$$

In the midplane $B_s = 0$ and thus $\vec{\nabla} \cdot \vec{B} = 0$.

Now

$$\vec{\nabla} \times \vec{B} = \left[\frac{1}{s}\cancelto{0}{\frac{\partial B_z}{\partial \phi}} - \frac{\partial B_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial B_s}{\partial z} - \cancel{\frac{\partial B_z}{\partial s}}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sB_\phi) - \cancel{\frac{\partial B_s}{\partial \phi}}\right]\hat{z}$$

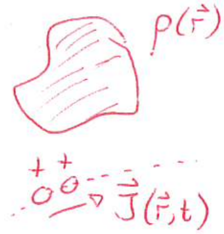
$$= -\frac{\partial B_\phi}{\partial z}\hat{s} + \frac{\partial B_s}{\partial z}\hat{\phi} + \frac{1}{s}\frac{\partial}{\partial s}(sB_\phi)\hat{z}$$

By typical Biot-Savart type arguments $B_s = 0$. However, we do not have a notion of how B_ϕ varies with z as we only computed B_ϕ for the $z=0$ plane. So we cannot directly check that this equation is valid.

Maxwell's Equations

We now have a complete system of equations for electromagnetism.

Sources



satisfy continuity equation

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

Sources are described by source densities:

$\rho(\vec{r},t)$

$\vec{J}(\vec{r},t)$

Sources produce fields that satisfy Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r},t)}{\epsilon_0}$$

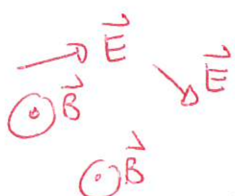
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

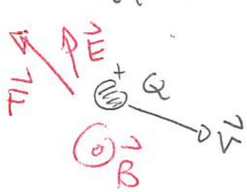
If sources are localized and fields $\to 0$ infinitely far away then these equations have unique solutions. Solve to get $\vec{E}, \vec{B}$

Sources produce fields



Fields exert forces on test charges

$$\vec{F} = Q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$



Lorentz force law