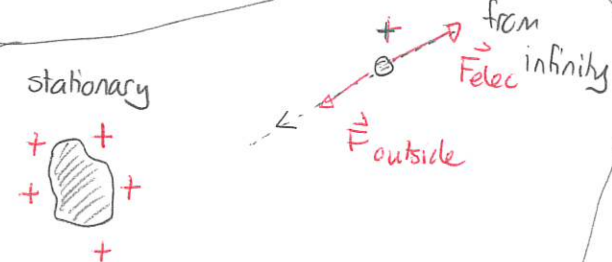
Weds: Read 7.3.1 → 7.3.3

Thurs: HW by 5pm

Energy and fields

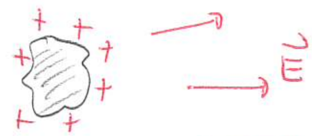
Recall that we were able to say that, for electrostatic electric fields, energy is stored in the field. This energy is defined to be the work needed to assemble the charge distribution, starting with point charges that are at rest infinitely far apart from each other.



Work done by outside force

$$W_{outside} = \int \vec{F}_{outside} \cdot d\vec{l}$$

→ Can be expressed in terms of the net electric field produced by all charges



Work required to assemble charges

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E}\cdot\vec{E}\, d\tau$$

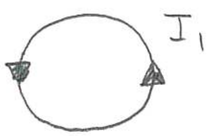
It is more convenient to compute this work using the electric field that the distribution ultimately produces rather than the definition involving force and displacement. We now consider a similar procedure in the context of assembling current distributions.

We will see that in order to assemble distributions of current we need an external force to do work, and this will appear in two places:

1) work required to assemble a single loop of current

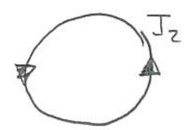
2) work required to bring multiple loops of current into each other's proximity.

loop 1



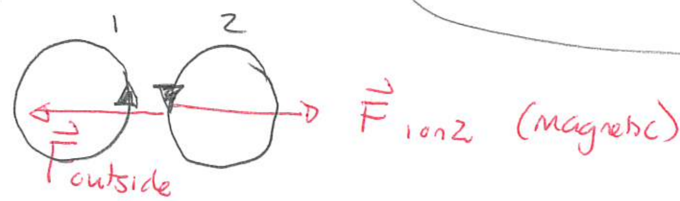
work is required to assemble this current

loop 2



work is required to assemble this current

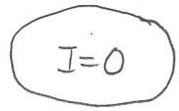
bring loops together requires extra work



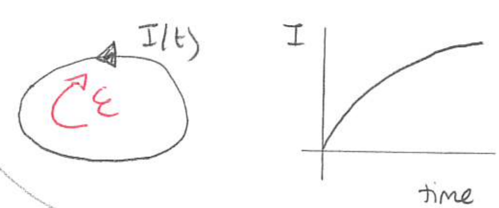
↳ needed to oppose magnetic ⇒ does work

First consider the process of assembling a current in a single loop.

Initially



Establish current, increases with time



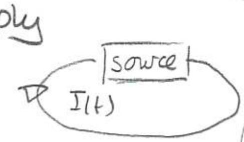
$\frac{dI}{dt} \neq 0$

Time varying current induces opposing EMF

$$\varepsilon = -L\frac{dI}{dt} \neq 0$$

Induced EMF opposes change in current, does work on current

↳ Requires outside source to supply energy needed to establish current



Consider the work done by a source in establishing a single loop of current. Let

$$W_{source}(t) = \text{work done from time } t=0 \text{ to time } t$$

Then the power delivered is

$$P(t) = \frac{dW_{source}}{dt} = \mathcal{E}_{source} I(t)$$

Now $\mathcal{E}_{source} = -\mathcal{E}_{induced}$ gives

$$P(t) = -\mathcal{E}_{induced} I(t)$$

$$= -\left[-L\frac{dI}{dt} I(t)\right] = L\,\frac{1}{2}\frac{d[I^2(t)]}{dt}$$

$$\Rightarrow \frac{dW_{source}}{dt} = \frac{d}{dt}\left[\frac{1}{2} L I^2\right]$$

Integrating gives:

$$W_{source} = \frac{1}{2} L I^2$$

This gives:

If there is initially zero current in a loop then the work needed to establish current I is $W = \frac{1}{2} L I^2$ where L is the self inductance of the loop.

1 Energy in a solenoid

A very long solenoid has radius R, length l, n turns per unit length and carries current I. The self-inductance of this is $L = \mu_0 \pi R^2 n^2 l$. Determine the energy stored in the current and express the result in terms of the magnetic field produced by the current and the volume of the solenoid.

Answer

$$W = \frac{1}{2} L I^2$$

$$= \frac{1}{2} \mu_0 \pi R^2 n^2 l \, I^2$$

The magnetic field within the solenoid is

$$B = \mu_0 n I$$

Thus

$$I = B / \mu_0 n$$

gives:

$$W = \frac{1}{2} \mu_0 \pi R^2 \cancel{n^2} l \frac{B^2}{\mu_0^{\cancel{2}} \cancel{n^2}} = \frac{1}{2\mu_0} \underbrace{\pi R^2 l}_{\text{volume}} B^2$$

Thus with volume $V = \pi R^2 l$ we get

$$W = \frac{1}{2\mu_0} B^2 V$$

This situation illustrates a general rule, which can be proved for any current arrangement.

> The work required to assemble a current distribution is
>
> $$W = \frac{1}{2\mu_0} \int_{\text{all space}} \vec{B} \cdot \vec{B} \, d\tau$$
>
> where $\vec{B}$ is the magnetic field produced by the current distribution

We define the energy stored in the magnetic field to be exactly the work required to assemble the current that produces the magnetic field.

Separately we define

> The energy density associated with the magnetic field is
>
> $$\frac{1}{2\mu_0} \vec{B} \cdot \vec{B}$$

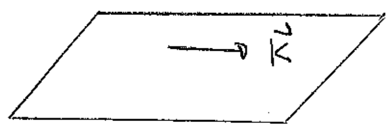
2 Energy stored between two parallel sheets of current

Two infinite parallel flat sheets are separated by distance d. They contain uniform constant currents, with surface current density K that flow in opposite directions. Determine the energy per unit volume in the area between the plates.

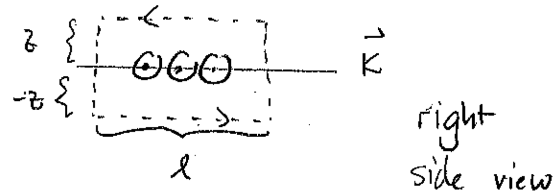
Answer: The energy per unit volume is

$$\frac{1}{2\mu_0}\vec{B}\cdot\vec{B}$$

and we need the field produced by the sheets.



$\approx$



$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \qquad \vec{K} = K\hat{y}$$

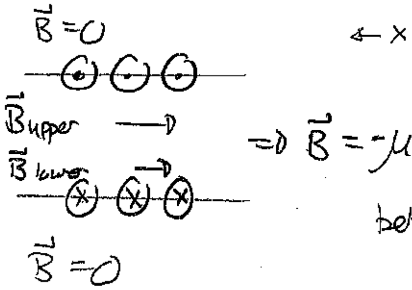
By Biot-Savart $B_y = 0$. By rotation $B_z = 0 \Rightarrow \vec{B} = B_x(z)\,\hat{x}$.
Then $B_x(-z) = -B_x(z)$. Using the Amperian loop illustrated above

$$\oint \vec{B}\cdot d\vec{l} = \int_0^l B_x(z)\,dx - \int_0^l B_x(-z)\,dx$$

$$= B_x(z)\,(2l) \qquad = \mu_0 I_{enc} \qquad = \mu_0 K l$$

$$\Rightarrow B_x(z) = \frac{\mu_0 K}{2}$$

For the pair of sheets



$$\Rightarrow \vec{B} = -\mu_0 K\,\hat{x}$$

between

energy per unit vol

$$\frac{1}{2\mu_0}\vec{B}\cdot\vec{B} = \boxed{\frac{\mu_0}{2}K^2}$$