Fri: HW by 5pm

Mon 30 Nov: Lecture ... Read 7.2.4, 7.3.1, 7.3.2.

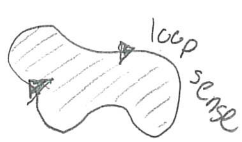
Faraday's Law

A general observation is that time-varying magnetic fields can produce electric fields. Faraday's Law quantifies this:

> Given a time-dependent magnetic field $\vec{B}$ there is an associated induced electric field which satisfies
>
> $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This can be converted to a flux rule:

Given $\vec{B}$ field and any closed loop



The magnetic flux through a surface that is bounded by the loop is

$$\Phi = \int_{\text{surface}} \vec{B} \cdot d\vec{a}$$

This is the same for any surface bounded by the same loop, provided $d\vec{a}$ and the loop sense are consistent

The induced electric field provides an EMF around the loop:

$$\mathcal{E} = \oint_{\text{loop}} \vec{E} \cdot d\vec{l}$$

Connected via

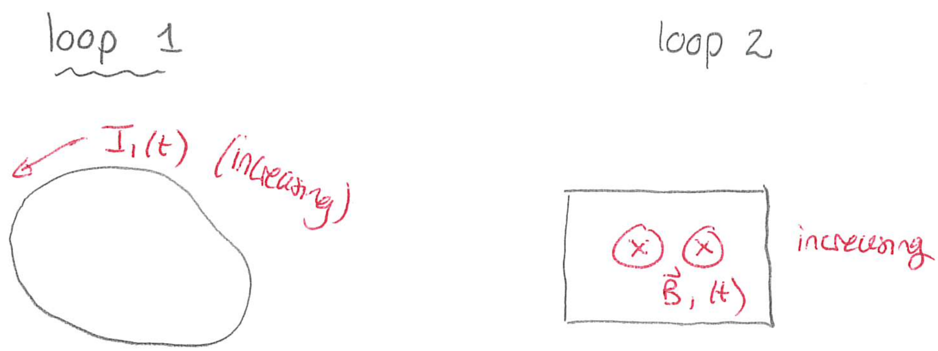
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Flux rule.

We can obtain the flux rule directly from Faraday's Law plus Stoke's theorem. A completely separate derivation shows that if the magnetic field remains constant and the loop configuration changes or moves then the same flux law is valid.

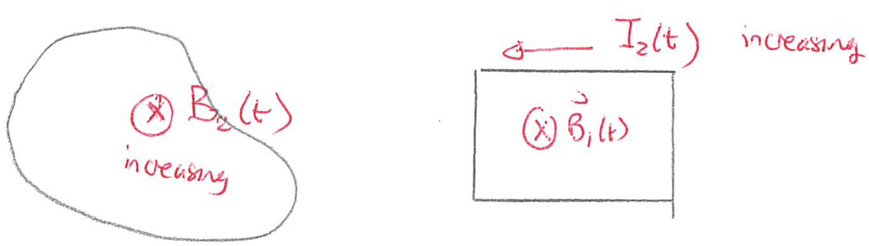
Inductance

We see that any time-varying current in a loop produces time varying fields and these produce currents in other loops. We could imagine an iterative mechanism such as:



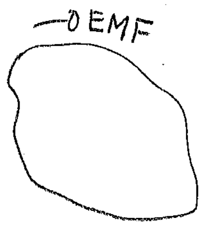
① Current in loop 1 produces time varying magnetic field $\vec{B}_1(t)$ and this exists within loop 2

② Time varying magnetic field in loop 2 induces an EMF around loop 2 and this produces a current around loop 2. For example:



This current produces its own magnetic field

③ This secondary magnetic field induces an EMF in the original loop and that affects the current in loop 1.



④ By such a back+forth process eventually two currents will become established in the two loops.

We let

$\mathcal{E}_1$ = EMF around loop 1

$\mathcal{E}_2$ = EMF " " 2

$I_1(t)$ = current in loop 1

$I_2(t)$ = " " " 2

$\vec{B}_1(t)$ = field produced by current in loop 1

$B_2(t)$ = field produced by $I_2(t)$

We aim to relate these. Rather than consider some sort of iterative process, we can analyze the overall situation simply in terms of the above parameters. We know that

$$\left.\begin{array}{l} \vec{B}_1(t) \text{ is proportional to } I_1(t) \\ \mathcal{E}_2 \text{ is proportional to } \vec{B}_1(t) \end{array}\right\} \Rightarrow \quad \mathcal{E}_2 \text{ is proportional to } I_1(t)$$

Thus we aim to show a relationship between $\mathcal{E}_2$ and $I_1(t)$ that demonstrates this proportionality. Similarly we should be able to find an analogous relationship between $\mathcal{E}_1$ and $I_2(t)$

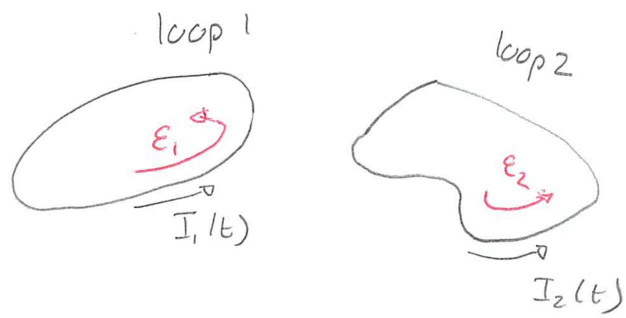
In general:

Let I_1 be the current in loop 1 and $\mathcal{E}_2$ the induced EMF in loop 2. Then

$$\mathcal{E}_2 = -M \frac{dI_1}{dt}$$

where M is called the mutual inductance and only depends on the configuration of the loops. Units of M: Henries (H)

Proof: Consider the two loops with indicated parameters.



Let

Φ_2 = flux through loop 2 as a result of current in loop 1

Then

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt}$$

and

$$\Phi_2 = \int_{\text{loop } 2} \vec{B}_1 \cdot d\vec{a}$$

where $\vec{B}_1$ is produced by $I_1(t)$. Now

$$\vec{B}_1 = \vec{\nabla} \times \vec{A}_1$$

where $\vec{A}_1$ is the vector potential for $\vec{B}_1$. We can choose it s.t. $\vec{\nabla} \cdot \vec{A}_1 = 0$

$$\vec{A}_1 = \frac{\mu_0}{4\pi} \oint_{\text{loop } 1} \frac{I_1}{r} d\vec{l}_1$$

Now

$$\Phi_2 = \int_{\text{loop }2} \vec{\nabla}\times\vec{A}_1 \cdot d\vec{a} = \oint_{\text{loop }2} \vec{A}_1 \cdot d\vec{l}_2$$

Thus

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$$

$$= I_1 \quad \frac{\mu_0}{4\pi} \oint_{\text{loop}1}\oint_{\text{loop}2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$$

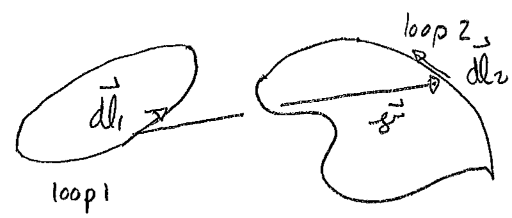
It follows that

$$\mathcal{E} = - M \frac{dI_1}{dt}$$

where

$$\boxed{M = \frac{\mu_0}{4\pi} \oint_{\text{loop}1}\oint_{\text{loop}2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}}$$

with the symbols as indicated



$\boxtimes$

The following fact is clear from the proof.

For any pair of loops the mutual inductance is the same regardless of which loop is producing the flux and which loop is experiencing the EMF. Thus

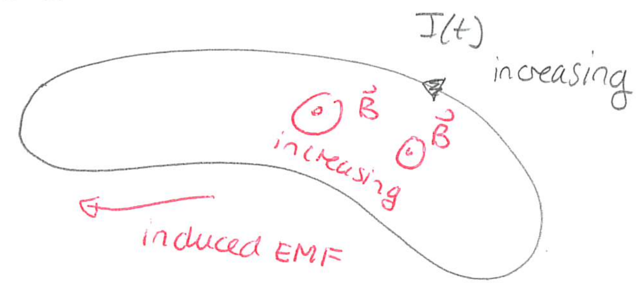
$$\mathcal{E}_2 = -M\frac{dI_1}{dt} \qquad \text{and} \qquad \mathcal{E}_1 = -M\frac{dI_2}{dt}$$

EMF induced in 2 by current in 1

EMF induced in 1 by current in 2

Self-inductance

Now consider a similar process for a single loop. A time varying current in this loop provides a time-varying magnetic field. The field produces a time-varying flux within the loop and that induces an EMF. We can see that the induced EMF "opposes" the change in the current.



We can follow a similar process. We find that, for a single loop, the flux is

$$\Phi = LI$$

where I is the current in the loop and L is the self-inductance of the loop. This again depends only on the loop configuration.

It then follows that

If a loop hosts a time-varying current $I(t)$ then this induces an EMF in the loop

$$\mathcal{E} = -L \frac{dI}{dt}$$

where L is the self-inductance of the loop.

1 Inductance in a solenoid

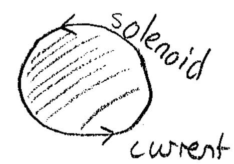
A very long solenoid has radius R, length l, n turns per unit length and carries current $I(t)$. If $l \gg R$ then one can assume that the solenoid is infinitely long for the purpose of determining magnetic fields.

a) Determine the magnetic flux through any single loop in the solenoid; this will be due to all other loops.

b) Determine and expression for the self-inductance of the solenoid.

c) Suppose that $R = 0.010\,\mathrm{m}$, $l = 0.20\,\mathrm{m}$ and $n = 1.0 \times 10^3\,\mathrm{turns/m}$. Determine the self-inductance of this solenoid.

Answer: a) Use $\Phi = LI$ where Φ is the flux through the shaded loop.



$$\Phi = \underbrace{\int}_{\text{surface}} \vec{B} \cdot d\vec{a}$$

For a solenoid

$$\vec{B} = \mu_0 n I \hat{z}$$

$$\left.\begin{array}{l} 0 \leq s' \leq R \\ 0 \leq \phi' \leq 2\pi \\ z' = \text{const} \end{array}\right\} d\vec{a} = s'ds'd\phi'\hat{z}$$

Thus $\vec{B} \cdot d\vec{a} = \mu_0 n I\, s'ds'd\phi'$

$$\Phi = \int_0^R ds' \int_0^{2\pi} d\phi' s' \mu_0 n I = \mu_0 n I \pi R^2$$

This is the flux through a single loop.

b) In length l there will be nl loops. So the total flux is

$$\Phi = nl \underbrace{\left[\mu_0 n I \pi R^2\right]}_{\text{single loop}} = I \mu_0 n^2 l \pi R^2$$

Using

$$\Phi = LI$$

we get

$$L = \mu_0 n^2 l \pi R^2$$

c)

$$L = 4\pi \times 10^{-7} \frac{Tm}{A} \times \left(1.0 \times 10^3/m\right)^2 0.20m \times \pi \times (0.010m)^2$$

$$= 0.80\,\pi^2 \times 10^{-5} \frac{Tm^2}{A}$$

$$= 7.9 \times 10^{-5}\ H$$