Fri: HW by 5pm

Mon: Read

Faraday's Law: Time-Varying Magnetic Fields

For both situations where a loop is moving through a stationary field and also when a stationary loop is placed in a time varying magnetic field:

1) there is a force per unit charge, $\vec{f}_s$, and this is independent of charge.

2) the EMF around any closed loop is

$$\mathcal{E} = \oint_{\text{loop}} \vec{f}_s \cdot d\vec{l}$$

3) the EMF satisfies

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where the magnetic flux is:

$$\Phi = \int_{\text{loop surface}} \vec{B} \cdot d\vec{a}$$

In cases where the loop is stationary and the field varies; Stoke's theorem gives

$$\vec{\nabla} \times \vec{f}_s = -\frac{\partial \vec{B}}{\partial t}$$

We then interpret $\vec{f}_s$ as an induced electric field $\vec{E}_{ind}$ and this does not arise from electrostatic sources. So

$$\vec{\nabla} \times \vec{E}_{ind} = -\frac{\partial \vec{B}}{\partial t}.$$

1 Decaying magnetic field

A magnetic field, given in cylindrical coordinates, decays according to

$$\mathbf{B} = \frac{k}{s} e^{-t/\tau}\,\hat{\mathbf{z}}$$

where $\tau > 0$ is a constant with units of time and $k > 0$ is also a constant.

a) Show that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$.

b) Determine a set of differential equations that the induced electric field must satisfy. Solve these so as to get a field of the form $\mathbf{E} = E_\phi \hat{\boldsymbol{\phi}}$. Find a solution that satisfies $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$.

c) Determine the EMF around any circular loop of radius R centered on the z axis.

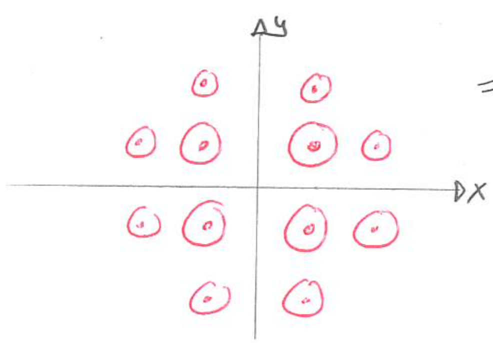
Answer: a) $\vec{\nabla} \cdot \vec{B} = \frac{1}{s}\frac{\partial}{\partial s}(s\overbrace{\cancel{B_s}}^{0}) + \frac{1}{s}\frac{\partial \overbrace{\cancel{B_\phi}}^{0}}{\partial \phi} + \frac{\partial B_z}{\partial z}$

$$= \frac{\partial}{\partial z}\left(\frac{k}{s}e^{-t/\tau}\right) = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

b) We will use (dropping the "ind" subscript)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

To get an insight consider a sketch of $\vec{B}$

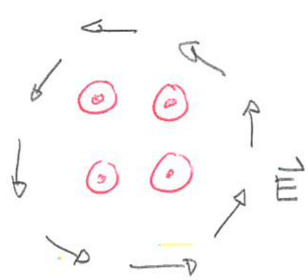


$\Rightarrow \frac{\partial \vec{B}}{\partial t}$ into page

$\Rightarrow -\frac{\partial \vec{B}}{\partial t}$ out of page

$\Rightarrow \vec{\nabla} \times \vec{E}$ out of page.

$\Rightarrow \vec{E}$ circles ??



So

$$\vec{\nabla}\times\vec{E} = \left[\frac{1}{s}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sE_\phi) - \frac{\partial E_s}{\partial \phi}\right]\hat{z}$$

$$= -\frac{\partial}{\partial t}\left(\frac{k}{s}e^{-t/\tau}\right)\hat{z}$$

$$= \frac{k}{\tau s}e^{-t/\tau}\,\hat{z}$$

$$\Rightarrow \frac{1}{s}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = 0$$

$$\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s} = 0$$

$$\cancel{\frac{1}{s}}\left[\frac{\partial}{\partial s}(sE_\phi) - \frac{\partial E_s}{\partial \phi}\right] = \frac{k}{\tau \cancel{s}}e^{-t/\tau}$$

We know that one of E_ϕ or E_s is non-zero. So we try $\vec{E} = E_\phi\hat{\phi}$

Thus:

$$\frac{\partial E_\phi}{\partial z} = 0 \qquad \Rightarrow \quad E_\phi = E_\phi(\phi, s)$$

$$\frac{\partial}{\partial s}(sE_\phi) = \frac{k}{\tau}e^{-t/\tau} \quad \Rightarrow \quad sE_\phi = \frac{k}{\tau}s\,e^{-t/\tau} + f(\phi)$$

$$\Rightarrow \quad E_\phi = \frac{k}{\tau}e^{-t/\tau} + \frac{1}{s}f(\phi)$$

Thus one possibility is

$$\vec{E} = \left[\frac{k}{\tau}e^{-t/\tau} + \frac{1}{s}f(\phi)\right]\hat{\phi}$$

Now consider

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{s}\frac{\partial}{\partial s}\overbrace{(s\cancel{E_s})}^{0} + \frac{1}{s}\frac{\partial E_\phi}{\partial \phi} + \overbrace{\frac{\partial \cancel{E_z}}{\partial z}}^{0}$$

$$= \frac{1}{s}\frac{\partial}{\partial \phi}\left[\frac{k}{\tau}e^{-t/\tau} + \frac{1}{s}f(\phi)\right]$$

Then $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \frac{1}{s^2}\frac{df}{d\phi} = 0 \Rightarrow f(\phi) = \alpha = \text{const}$

In this case

$$\vec{E} = \left[\frac{k}{\tau} e^{-t/\tau} + \frac{\alpha}{s}\right]\hat{\phi}$$

We now have a situation where the fields satisfy:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

Also $\vec{\nabla}\times\vec{B} = -\frac{\partial B_z}{\partial s}\hat{\phi} = \frac{k}{s^2}e^{-t/\tau}\hat{\phi}$ and so the magnetic field could be produced by a time varying current (with eventually a contribution from a time-varying electric field).

Note that $\alpha \neq 0$ describes a line of charge since

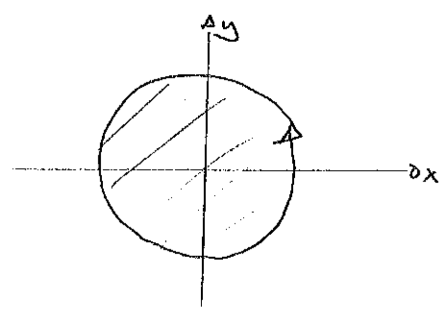
$\vec{\nabla}\cdot\left(\frac{1}{s}\hat{\phi}\right) = \frac{1}{s}\delta(s)\hat{z}$

c) Consider the illustrated loop:

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\Phi = \int_{\text{surface}} \vec{B}\cdot d\vec{a}$$

$$\begin{cases} 0 \le s' \le R \\ 0 \le \phi' \le 2\pi \\ z' = \text{const} \\ d\vec{a} = s'ds'd\phi'\hat{z} \end{cases}$$



$$\vec{B}\cdot d\vec{a} = \frac{k}{\cancel{s}}e^{-t/\tau}\cancel{s'}ds'd\phi' = ke^{-t/\tau}ds'd\phi'$$

$$\Rightarrow \Phi = \int_0^R ds'\int_0^{2\pi} d\phi' ke^{-t/\tau} = 2\pi Rke^{-t/\tau}$$

$$\frac{\partial\Phi}{\partial t} = -\frac{2\pi Rk}{\tau}e^{-t/\tau} \Rightarrow \mathcal{E} = \frac{2\pi Rk}{\tau}e^{-t/\tau}$$

and this is positive c.c.w.

Field equations for electric fields

It appears that there are two types of electric field

	Electrostatic field	Field induced by magnetic field
Source	stationary charges $\rho(\vec{r})$ 	time varying magnetic field 
Field satisfies	$\vec{\nabla} \cdot \vec{E}_{elec} = \rho/\epsilon_0$ $\vec{\nabla} \times \vec{E}_{elec} = 0$	$\vec{\nabla} \cdot \vec{E}_{ind} = 0$ $\vec{\nabla} \times \vec{E}_{ind} = -\frac{\partial \vec{B}}{\partial t}$

We can then redefine the electric field as

$$\vec{E} = \vec{E}_{elec} + \vec{E}_{ind}$$

and this satisfies

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho/\epsilon_0 \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}. \end{aligned}}$$

Computing induced fields.

For an induced electric field

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

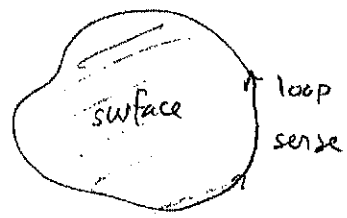
which are similar to

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Mathematical techniques for determining $\vec{E}$ are analogous to those for determining magnetostatic fields. For example consider a closed loop. Then

$$\oint_{loop} \vec{E}\cdot d\vec{l} = \int_{surface} \vec{\nabla}\times\vec{E}\cdot d\vec{a}$$



$$= -\int_{surface} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$= -\frac{\partial \Phi}{\partial t}$$

This is analogous to Ampère's law with $\mu_0 I_{enc} \to -\frac{\partial \vec{B}}{\partial t}$

Similarly

	$\vec{B}$ field calculation	$\vec{E}$ field calculation
source	$\vec{J}$	$-\frac{\partial \vec{B}}{\partial t}$
field equations	$\vec{\nabla}\cdot\vec{B}=0$ $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$	$\vec{\nabla}\cdot\vec{E}=0$ $\vec{\nabla}\times\vec{E}=-\frac{\partial \vec{B}}{\partial t}$
Biot-Savart	$\vec{B}=\frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$	$\vec{E}=-\frac{1}{4\pi}\int \frac{\frac{\partial \vec{B}}{\partial t}\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$
Ampère :	$\oint_{loop}\vec{B}\cdot d\vec{l}=\mu_0 \underbrace{I_{enc}}_{\int_{surface}\vec{J}\cdot d\vec{a}}$	$\oint_{loop}\vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_{surface}\vec{B}\cdot d\vec{a}$

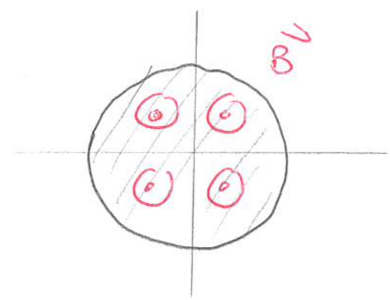
2 Induced electric field

Consider a magnetic field restricted to a cylindrical region with radius R. In cylindrical coordinates,

$$\mathbf{B} = \begin{cases} B(t)\,\hat{\mathbf{z}} & \text{if } s \leqslant R \\ 0 & \text{if } s \geqslant R \end{cases}$$

where $B(t)$ only depends on time.

a) Describe the direction of the induced electric field.
b) Determine an expression for the induced electric field.
c) Evaluate $\nabla \cdot \mathbf{E}$ and $\nabla \cdot \mathbf{B}$.

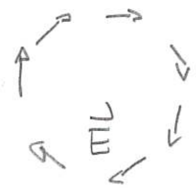


Answers: a) We have

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

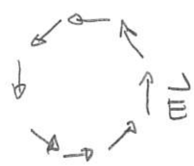
Then if $\vec{B}$ increases

$\vec{\nabla} \times \vec{E}$ is $\otimes$ $\Rightarrow$



and if $\vec{B}$ decreases

$\vec{\nabla} \times \vec{E}$ is $\odot$ $\Rightarrow$



b) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and here $\frac{\partial \vec{B}}{\partial t}$ serves as a source current.

Using any loop

$$\int_{\text{surface}} \vec{\nabla} \times \vec{E} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a}$$

$$\Rightarrow \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a}$$

By symmetry $\vec{E} = E_\phi(s)\,\hat{\phi}$ and with a circular loop of radius s

$$\int \vec{E}\cdot d\vec{l} = \int E_\phi(s)\, s'\, d\phi' = E_\phi(s)\, s\, 2\pi = -\frac{d}{dt}\int_{\text{surface}} \vec{B}\cdot d\vec{a}$$

Then if $s<R$

$$\int_{\text{surface}} \vec{B}\cdot d\vec{a} = \int_0^s ds' \int_0^{2\pi} d\phi' s'\, B(t) = \pi s^2 B(t)$$

Thus: $\cancel{2}\cancel{\pi}\cancel{s}\, E_\phi(s) = -\cancel{\pi} s^{\cancel{2}} \frac{dB}{dt}$

$\Rightarrow \quad E_\phi(s) = -\frac{s}{2}\frac{dB}{dt} \qquad \Rightarrow \quad \vec{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\phi}$ is $s<R$

If $s>R$ $\quad \int \vec{B}\cdot d\vec{a} = \pi R^2 B(t) \qquad \Rightarrow \qquad \vec{E} = -\frac{R^2}{2s}\frac{dB}{dt}\hat{\phi}$

c) $\vec{\nabla}\cdot\vec{B} = 0$ trivially

$\vec{\nabla}\cdot\vec{E} = 0$ by inspection