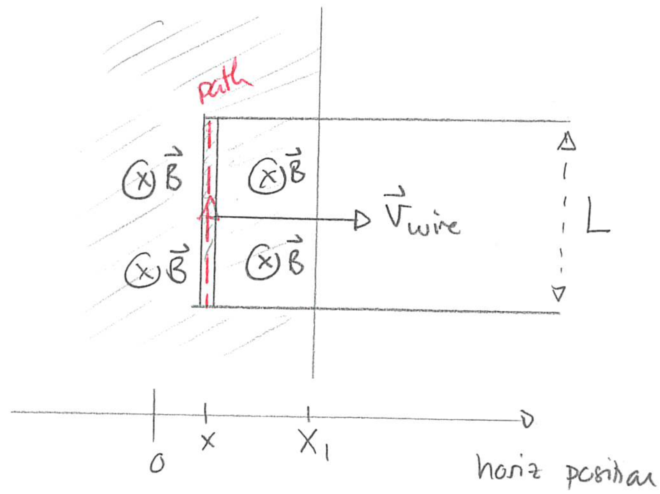
Phys 311 2020

Lecture 37

Weds: Read

Fri: HW

Motional EMF



Consider a loop that is dragged through a uniform magnetic field as illustrated. We saw that this motion results in forces acting on mobile charges in the loop. In this case the force per unit charge is

$$\vec{f}_s = B\, v_{wire}\, \hat{y}$$

The EMF along the indicated path is $(d\vec{l} = dy\,\hat{y})$

$$\mathcal{E} = \int \vec{f}_s \cdot d\vec{l} = \int B v_{wire}\, dy \Rightarrow \mathcal{E} = B v_{wire} L$$

Then $v_{wire} = \frac{dx}{dt}$ and thus

$$\mathcal{E} = B \frac{dx}{dt} L$$

Now the area of the field within the loop is $A = L(X_1 - x)$. Thus

$$\frac{dx}{dt} = -\frac{dA}{dt} \Rightarrow \mathcal{E} = -B\frac{dA}{dt} = -\frac{d}{dt}(BA)$$

We will show that a rule of this type is true for any loop that is dragged through a magnetic field which is time-independent. However, the rule will be stated in terms of magnetic flux, which we now introduce.

Magnetic flux

We define magnetic flux via:

> Let $\vec{B}$ be a magnetic field. Consider any loop. Then the magnetic flux through the loop is
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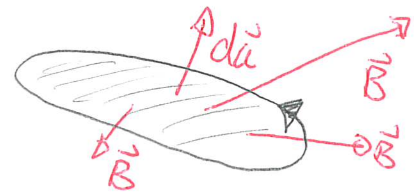
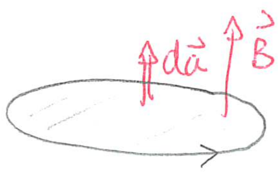
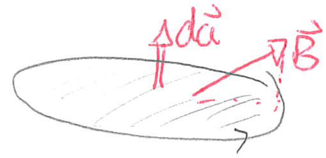
> 
>
>
> $$\Phi = \int_S \vec{B}\cdot d\vec{a}$$
>
> where S is any surface bounded by the loop. Units: Weber (Wb)

Conceptually the magnetic flux quantifies the extent to which the field penetrates the surface



larger Φ

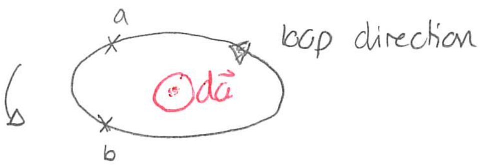


smaller Φ

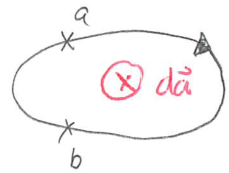
There are various issues that we need to consider before using flux

1) <u>direction of area vector</u>

The direction of the area vector is established by using the current direction in conjunction with the loop direction rule.



EMF positive $\Rightarrow$ $V_b - V_a > 0$



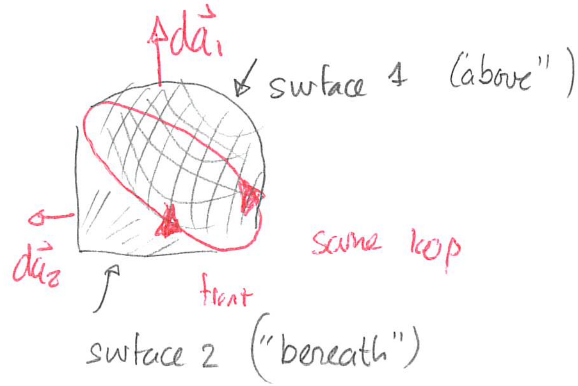
EMF negative $\Rightarrow$ $V_a - V_b < 0$

$\hookrightarrow$ reversing loop direction reverses EMF but $V_b - V_a$ sign stays same

2) flux is well defined for any surface

Given any loop, there are multiple surfaces that bound it. These can be shown to yield the same flux provided that $\vec{\nabla} \cdot \vec{B} = 0$.

Consider the illustrated case. Then



$$\underset{\text{surface 1}}{\oint} \vec{B} \cdot d\vec{a}_1 + \underset{\text{surface 2}}{\oint} \vec{B} \cdot d\vec{a}_2 = \underset{\text{closed surface}}{\oint} \vec{B} \cdot d\vec{a}$$

$$= \int_{\text{region}} \vec{\nabla} \cdot \vec{B} \, d\tau$$

$$= 0$$

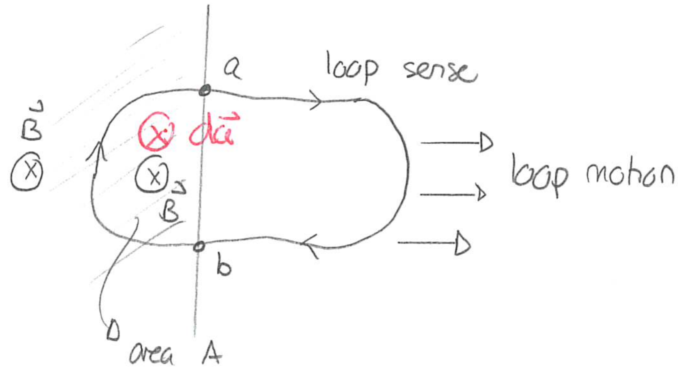
$$\Rightarrow \int_{\text{surface 1}} \vec{B} \cdot d\vec{a}_1 + \int_{\text{surface 2}} \vec{B} \cdot d\vec{a}_2 = 0$$

Now reversing the orientation of the area vector on surface 2 gives $d\vec{a} = -d\vec{a}_2$. The orientation of $d\vec{a}$ is consistent with loop direction rules. So is $d\vec{a} = d\vec{a}_1$. Then:

$$\int_{\text{surface 1}} \vec{B} \cdot d\vec{a} - \int_{\text{surface 2}} \vec{B} \cdot d\vec{a} \Rightarrow \int_{\text{surface 1}} \vec{B} \cdot d\vec{a} = \int_{\text{surface 2}} \vec{B} \cdot d\vec{a} \qquad \blacksquare$$

3) Positive /negative EMF

Recall that if we use the moving loop as a driving source for a current then, using the illustrated loop sense,



$$\mathcal{E} + V_b - V_a = 0$$

So

$$\mathcal{E} = V_a - V_b$$

This tells us where $V_a > V_b$ or the other way round. In this case, for a uniform field

$$\Phi = BA$$

and

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Since $\frac{d\Phi}{dt} < 0$ we get $\mathcal{E} > 0 \Rightarrow V_a > V_b$ in this case.

If we reversed the loop sense we would get

Φ reverses sign $\Rightarrow$ $\mathcal{E}$ reverses sign $\Rightarrow$ $V_b - V_a$ reverses sign

$\Rightarrow$ $V_a - V_b$ same sign.

So this is consistent.

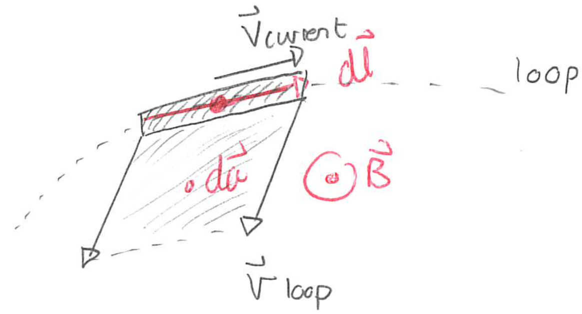
We showed this for a special configuration of a loop. We can extend this to non-rectangular loops and non-uniform fields. We can show that

> For any loop moving through a time-independent magnetic field $\vec{B}$ there is an induced force per unit charge, $\vec{f}_s$, that is independent of the current carriers. This produces an EMF
>
> $$\mathcal{E} = \int_{\text{loop}} \vec{f}_s \cdot d\vec{l}$$
>
> which satisfies
>
> $$\mathcal{E} = -\frac{d\Phi}{dt}$$
>
> where the flux through the loop is
>
> $$\Phi = \int_S \vec{B} \cdot d\vec{a}$$
>
> and S is any surface enclosed by the loop. → direction rules apply.

<u>Proof</u>: Consider a portion of the loop that moves as illustrated:

Then the force on any charge in this section is:

$$\vec{F} = q\left(\vec{v}_{loop} + \vec{v}_{current}\right) \times \vec{B}$$



⇒ there exists a force per unit charge

$$\vec{f}_s = \left(\vec{v}_{loop} + \vec{v}_{current}\right) \times \vec{B}$$

Now the EMF contribution from this section is:

$$\vec{f}_s \cdot \vec{dl} = (\vec{V}_{loop} \times \vec{B}) \cdot \vec{dl} + (\vec{V}_{current} \times \vec{B}) \cdot \vec{dl}$$

Then $\vec{v}_{current}$ and $\vec{dl}$ are parallel $\Rightarrow$ $(\vec{V}_{current} \times \vec{B}) \cdot \vec{dl} = 0$

$$\Rightarrow \vec{f}_s \cdot \vec{dl} = \vec{V}_{loop} \times \vec{B} \cdot \vec{dl}$$

$$= \vec{dl} \cdot (\vec{v}_{loop} \times \vec{B}) = \vec{B} \cdot (\vec{dl} \times \vec{V}_{loop}) = - \vec{B} \cdot (\vec{v}_{loop} \times \vec{dl})$$

Now geometry shows that

$$[\vec{V}_{loop} \times \vec{dl}]\, dt = d\vec{a}$$

where $d\vec{a}$ is the area vector for the shaded portion. So

$$\vec{f}_s \cdot \vec{dl} = - \vec{B} \cdot d\vec{a} / dt = - \vec{B} \cdot \frac{d\vec{a}}{dt}$$

$$= - \frac{d}{dt} \vec{B} \cdot d\vec{a}$$

since $\vec{B}$ is time independent. Then we get that

$$\mathcal{E} = - \frac{d}{dt} \int_{\text{loop surface}} \vec{B} \cdot d\vec{a} \quad \Rightarrow \quad \mathcal{E} = - \frac{d\Phi}{dt}$$

for the entire loop. ∎

1 Rotating loop in a magnetic field

A circular loop with radius R initially lies in the xy plane (with its axis along the z axis). This is in the presence of a constant external magnetic field,

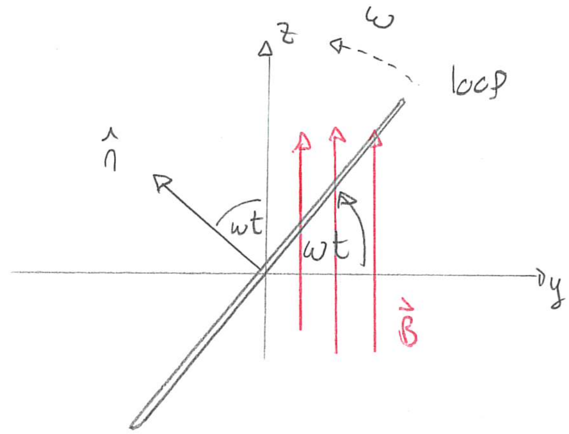
$$\mathbf{B} = \begin{cases} 0 & \text{if } y \leqslant 0 \\ B\hat{\mathbf{z}} & \text{if } y \geqslant 0 \end{cases}$$

where $B > 0$. The loop rotates with constant angular velocity ω counterclockwise about the x axis. Determine the EMF at all times.

Answer: Here

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Side View



and

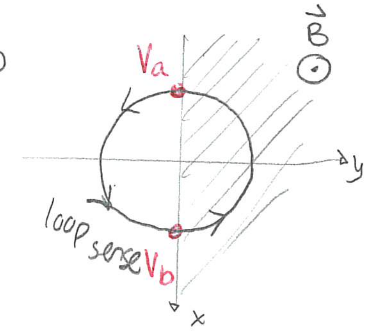
$$\Phi = \int \vec{B}\cdot d\vec{a}$$

Then

$$d\vec{a} = da\,\hat{n}$$

and $\hat{n} = \cos\omega t\,\hat{z} - \sin\omega t\,\hat{y}$

top view



$$\Rightarrow \vec{B}\cdot d\vec{a} = \begin{cases} 0 & y \leq 0 \\ B\cos\omega t & y \geq 0 \end{cases}$$

$$\Rightarrow \Phi = \int_{\text{half loop}} B\cos\omega t\, da = B\cos\omega t\,\frac{\pi R^2}{2}$$

So

$$\mathcal{E} = -\frac{d}{dt}\frac{\pi R^2}{2}B\cos\omega t = \frac{\pi R^2 B}{2}\omega\sin\omega t$$

If the loop rotates as picture then at this moment $\mathcal{E} > 0 \Rightarrow V_a > V_b$

$\Rightarrow$ current flows in loop sense $\Rightarrow$ C.C.W

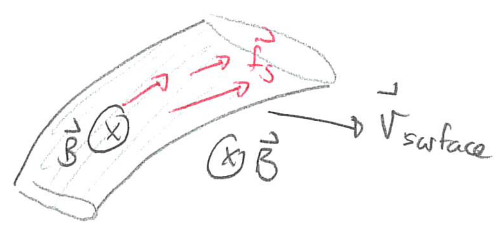
Faraday's Law

We have seen that for a moving loop, there is a non-electrostatic force per unit charge associated with motion in a magnetic field $\vec{f}_s$ and that this satisfies

$$\mathcal{E} = -\frac{d\Phi}{dt} \Rightarrow \underset{\text{closed loop}}{\int} \vec{f}_s \cdot d\vec{l} = -\frac{d}{dt} \underset{\text{loop surface}}{\int} \vec{B} \cdot d\vec{a}$$

For motional EMF the force will only exist within the loop. However we could extend it to three dimensional objects dragged through the field. Then within the object

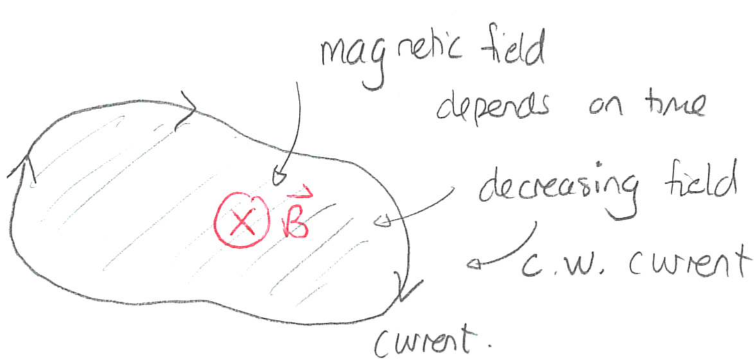
$$\underset{\text{loop}}{\int} \vec{f}_s \cdot d\vec{l} = \underset{\text{surface}}{\int} \vec{\nabla} \times \vec{f}_s \cdot d\vec{a}$$



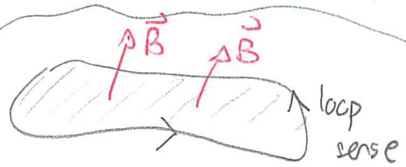
will give:

$$\int \vec{\nabla} \times \vec{f}_s \cdot d\vec{a} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a}$$

This suggest that we consider situations where the loop is fixed but the field varies with time. Expeimental observations indicate that this also produces a current or a voltage difference.

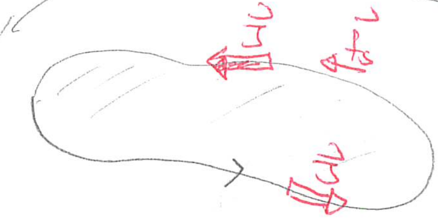


This also cannot be described via electrostatic fields (at least for Ohmic materials). Experimental evidence supports the following.



Time-varying magnetic field

$\longrightarrow$



Flux $\Phi = \int \vec{B}\cdot d\vec{a}$ varies with time. Current is observed to flow

$\downarrow$

If loop is Ohmic time-varying flux produces EMF (voltage difference)

$\longleftarrow$

There is an nonelectrostatic force per unit charge $\vec{f_s}$ that takes the place of $\vec{E}$ in Ohm's law $\vec{J} = \sigma\vec{E} = \sigma\vec{f_s}$

Now, if the loop configuration is fixed then we get

$$\frac{d}{dt}\Phi = \frac{d}{dt}\int \vec{B}\cdot d\vec{a} = \int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

and the evidence is that

$$\mathcal{E} = \int \vec{\nabla}\times\vec{f_s}\cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

is true for all loops. Thus the force per unit charge satisfies

$$\vec{\nabla}\times\vec{f_s} = -\frac{\partial \vec{B}}{\partial t}$$

This gives Faraday's Law for a fixed loop in a time-varying magnetic field.

Any time-varying magnetic field generates a non-electrostatic force per unit charge that satisfies:

$$\vec{\nabla} \times \vec{f}_s = -\frac{\partial \vec{B}}{\partial t}$$

Since the force that this exerts on a charge q is $\vec{F} = q\vec{f}_s$ it plays the role of an electric field. We denote this $\vec{E}_{ind}$ for the induced electric field. So

Any time-varying magnetic field induces a non-electrostatic electric field $\vec{E}_{ind}$ that satisfies:

$$\vec{\nabla} \times \vec{E}_{ind} = -\frac{\partial \vec{B}}{\partial t}$$