Tues: HW

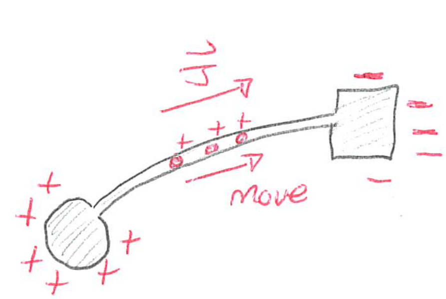
Weds: Read

Fri: Test

Current + charge densities

We assume that currents consist of moving charged particles and that charge is conserved. Then it follows that the charge density ρ and current density $\vec{J}$ must satisfy the continuity equation:

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$



Note that this simply provides one constraint and does not mean that, if we know one of the two densities we can determine the other. For example if ρ satisfies the constraint and ρ_0 is independent of time then

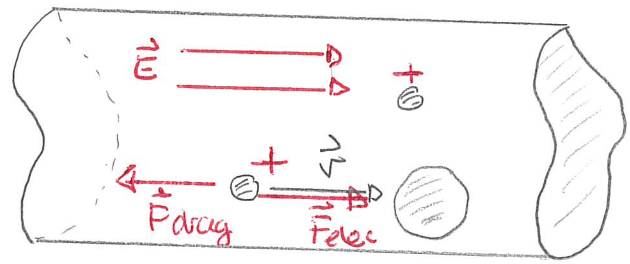
$$\frac{\partial(\rho+\rho_0)}{\partial t} = \frac{\partial \rho}{\partial t}$$

means that $\rho+\rho_0$ also satisfies the constraint.

The steady-state (or stationary) situations require that ρ and $\vec{J}$ are independent of time. Here then $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{J} = 0$ and $\frac{\partial \vec{J}}{\partial t} = 0$

Steady currents in matter: Ohm's Law

We now consider a steady current in a material. In general the moving charges will collide with stationary particles in the material and these collisions will impede the motion of such moving charges.



This resistance would result in a reduction of the current over time. We know that it is possible to sustain a current and thus seek a model for this

Demo: PhET Battery - Resistor

Assume that the moving charges are positive (in reality they are usually negative but the same argument holds). Let $\vec{v}$ be the velocity of a moving charge. We then assume that the impedance can be modeled by a drag force

$$\vec{F}_{drag} = -\alpha\vec{v}$$

where $\alpha > 0$ is a constant that depends on the material. We can attain a steady flow via an electrostatic field $\vec{E}$ that produces a force $\vec{F}_{elec} = q\vec{E}$. Then

$$\vec{F}_{elec} + \vec{F}_{drag} = m\vec{a} \quad \Rightarrow \quad q\vec{E} - \alpha\vec{v} = m\vec{a}$$

Over time the acceleration averages to zero. Denoting an average over time by $\langle \cdots \rangle$ we get

$$\langle q\vec{E} - \alpha\vec{v} \rangle = m\langle \vec{a} \rangle = 0 \quad \Rightarrow \quad \langle \vec{v} \rangle = \frac{q}{\alpha}\vec{E}$$

If the density of the charges that carry the current is ρ then

$$\vec{J} = \rho \langle \vec{v} \rangle$$

$$\Rightarrow \vec{J} = \frac{\rho q}{\alpha} \vec{E}$$

Suppose that each particle in the current has charge q and that the number of these per unit volume is n. Then

$$\vec{J} = \frac{n q^2}{\alpha} \vec{E}$$

We see that the current density is proportional to the electrostatic field that is required to sustain the current.

This is an example of Ohm's law which holds for Ohmic materials.

> In an Ohmic material the current density $\vec{J}$ and electric field $\vec{E}$ are related by
>
> $$\vec{J} = \sigma \vec{E}$$
>
> where σ is a constant called the conductivity Units: $(\Omega m)^{-1}$

The conductivity only depends on the material. Not all materials obey this law and it must be checked experimentally.

In a more detailed model if τ is the time between collisions and m the mass of the charge then classical mechanics gives $\alpha = m/\tau$

Thus

$$\sigma = \frac{n q^2}{m} \tau = \frac{n e^2}{m_e} \tau \quad \text{(for electrons)}$$

This is the Drude model.

To summarize

For an Ohmic material with a steady state current described by current density $\vec{J}$, there is an electric field $\vec{E}$ and inside the material

i) $\vec{J} = \sigma \vec{E}$ (Ohm's Law)

ii) $\vec{\nabla} \cdot \vec{J} = 0$ (steady state)

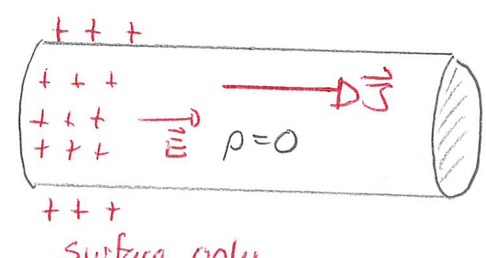
iii) $\vec{\nabla} \cdot \vec{E} = 0$ (combination of i, ii)

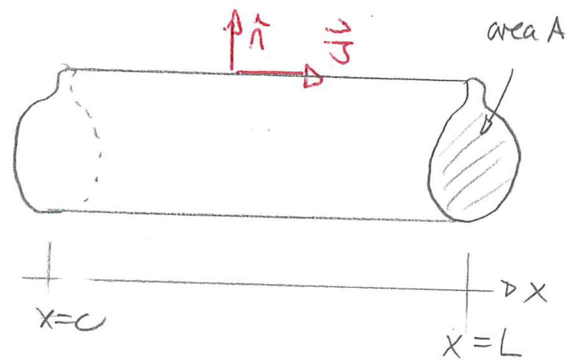
iv) $\rho = 0$ (Gauss' Law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$)

On the surface it is possible that $\rho \neq 0$

So excess charge will reside on the surface only. In this case $\partial\rho/\partial t = 0$

A typical situation which one can consider is that where the Ohmic material has a cylindrical cross-section. We assume that no charge flows out through the curve surface. So if $\hat{n}$ is perpendicular to the surface then $\hat{n} \cdot \vec{J} = 0$ at the surface. We want to address questions such as:





1) what the potential throughout the material is

2) what the resistance is.

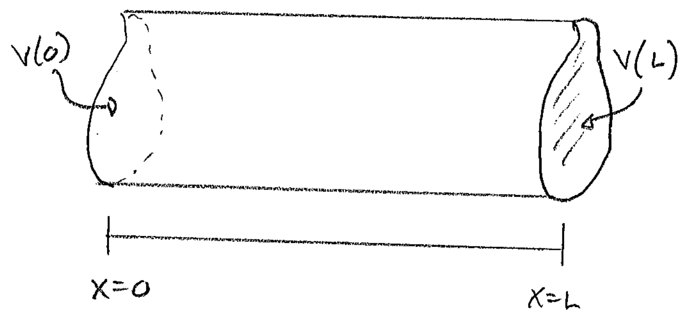
Now we have that inside the material:

$$\vec{\nabla} \cdot \vec{J} = 0$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{E} = 0$$

But $\vec{E} = -\vec{\nabla} V \quad \Rightarrow \quad \nabla^2 V = 0$ inside the material

In order to solve this we need to apply boundary conditions. On the curved surface



$$\vec{J} = -\sigma \vec{\nabla} V$$

But $\hat{n} \cdot \vec{J} = 0$

$\Rightarrow \quad \hat{n} \cdot \vec{\nabla} V = 0$ on curved surface

Finally we assume that the potential at $x=0$ is uniform and similarly at $x=L$. With these two boundary conditions there is only one solution to Laplace's equation: The unique solution is:

$$\boxed{V(x) = \frac{V(L) - V(0)}{L} x + V(0)}$$

Then we can check:

1) $\nabla^2 V = 0$ inside

2) $V(0) = V(0)$ and $V(L) = V(L)$

3) $\vec{\nabla} V = \frac{V(L) - V(0)}{L} \hat{x}$ inside $\Rightarrow$ $\hat{n} \cdot \vec{\nabla} V = 0$ on surface.

1 Ohm's law

Consider a cylindrical material with arbitrary cross section. Suppose that this obeys $\mathbf{J} = \sigma \mathbf{E}$ where the conductivity σ is independent of location within the material.

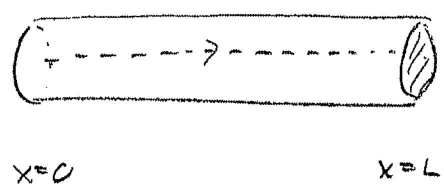
a) By using a path along the length of the cylinder and the relationship between ΔV and $\mathbf{E}$, show that the potential difference across the ends satisfies

$$\Delta V = IR$$

where I is the current along the material and R a constant.

b) Determine an expression for R in terms of the conductivity, the cross sectional area of the material and the length of the material.

Answer: a)



Consider the illustrated path. Then

$$\Delta V = -\int \vec{E}\cdot d\vec{l}$$

$$\Rightarrow \quad V(L) - V(0) = -\int \vec{E}\cdot d\vec{l}$$

Now $\vec{E} = \frac{1}{\sigma}\vec{J}$ means

$$V(L) - V(0) = -\frac{1}{\sigma}\int \vec{J}\cdot d\vec{l}$$

We now show J is uniform across the cross-section.

$$\vec{E} = -\vec{\nabla} V = -\frac{V(L)-V(0)}{L}\,\hat{x} \text{ is uniform} \Rightarrow \vec{J} = \sigma\vec{E} \text{ is uniform}$$

So $\vec{J} = J\hat{x}$. Thus

$$V(L) - V(0) = -\frac{J}{\sigma}\int \hat{x}\cdot d\vec{l} = -\frac{JL}{\sigma} \Rightarrow \Delta V = V(0) - V(L) = \frac{JL}{\sigma}$$

Now $J = I/A \Rightarrow \Delta V = \frac{L}{\sigma A} I$

b) $R = \dfrac{L}{\sigma A}$

The example illustrates:

For an Ohmic material with a cylindrical cross-section and a current constrained so that it does not flow out of the cylinder sides, in a steady state; the potential difference between two points separated by length L is

$$\Delta V = IR$$

where I is the current that flows and

$$R = \frac{L}{\sigma A}$$

is the resistance.

We define the resistivity as

$$\rho = 1/\sigma$$

and then

$$R = \frac{\rho L}{A}$$

Power and electric currents

The electric field exerts an electric force on each moving charge. Thus the field does non-zero work on each charge in order to sustain the current. The power delivered in this way is the rate at which work is done. We aim to determine a general expression for this and then use it to determine power delivered in Ohmic materials.

Classical mechanics states that:

> The power delivered by force $\vec{F}$ that acts on an object moving with velocity $\vec{v}$ is
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> $$P = \vec{F} \cdot \vec{v}$$
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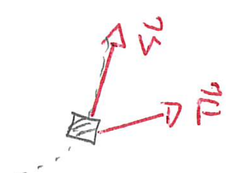
> 
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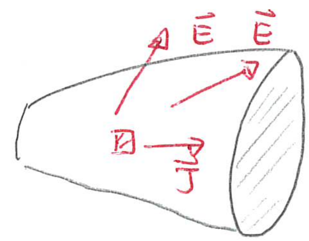
Then we can prove:

> If a current with current density $\vec{J}$ flows in the presence of an electric field $\vec{E}$, the power delivered by the field in a region R is
>
> $$P = \int_R \vec{E} \cdot \vec{J} \, d\tau$$
>
> 
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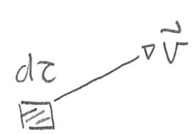
Proof: Consider a small portion with volume $d\tau$

Let

$\vec{v}$ = velocity at this point

ρ = charge density of moving charges



↳ charge $dq = \rho \, d\tau$

Then with , $dq = \rho\, d\tau$, the force exerted by the field is

$$d\vec{F} = dq\, \vec{E}$$

and the power delivered is

$$\begin{aligned} dP &= dq\, \vec{E} \cdot \vec{v} \\ &= \vec{E} \cdot dq\, \vec{v} \\ &= \vec{E} \cdot \rho \vec{v}\, d\tau \\ &= \vec{E} \cdot \vec{J}\, d\tau . \end{aligned}$$

Integrating gives

$$P = \int_R \vec{E} \cdot \vec{J}\, d\tau$$

∎

Then for an <u>Ohmic material</u> $\vec{E} = \vec{J}/\sigma$ gives.

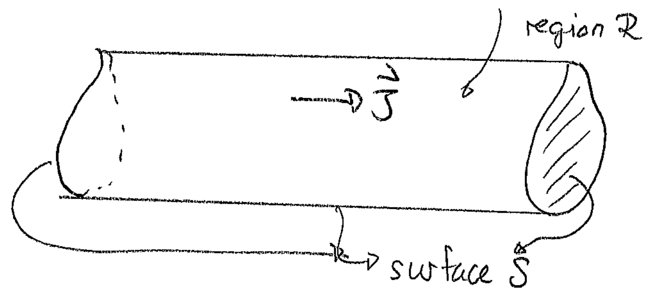
$$\boxed{P = \frac{1}{\sigma} \int_R \vec{J} \cdot \vec{J}\, d\tau}$$

This is always positive and so in an Ohmic material the field always supplies energy.

We can use the general result to show:

For an Ohmic material with a cylindrical cross section where current cannot leave the sides $$P = I\,\Delta V$$ where ΔV is the potential difference across the ends.

<u>Proof:</u> $$P = \int_R \vec{E}\cdot\vec{J}\,d\tau$$



But

$$\vec{E} = -\vec{\nabla}V$$

$$\Rightarrow P = -\int(\vec{\nabla}V)\cdot\vec{J}\,d\tau$$

Then

$$\vec{\nabla}\cdot(V\vec{J}) = (\vec{\nabla}V)\cdot\vec{J} + V\,\vec{\nabla}\cdot\vec{J}$$

$$\Rightarrow P = -\int_R \vec{\nabla}\cdot(V\vec{J})\,d\tau + \int V(\vec{\nabla}\cdot\vec{J})\,d\tau$$

In a steady state $\vec{\nabla}\cdot\vec{J} = 0$ and thus

$$P = -\int_R \vec{\nabla}\cdot(V\vec{J})\,d\tau$$

$$= -\oint_{\text{surface}} (V\vec{J})\cdot d\vec{a}$$

$$\Rightarrow \qquad P = -\int_{\text{left end}} V\vec{J}\cdot d\vec{a} \;-\; \int_{\text{right end}} V\vec{J}\cdot d\vec{a} \;+\; \int_{\text{sides}} V\vec{J}\cdot d\vec{a}$$

Then $\vec{J}\cdot d\vec{a} = 0$ on the sides gives:

$$P = -\int_{\text{left}} V\,\vec{J}\cdot d\vec{a} \;-\; \int_{\text{right}} V\,\vec{J}\cdot d\vec{a}$$

At this point we assume that the potential at each end is fixed. Then

$$P = -V_{left}\int_{\text{left}} \vec{J}\cdot d\vec{a} \;-\; V_{right}\int_{\text{right}} \vec{J}\cdot d\vec{a}$$

$= -V_{left}$ (current leaving left side) $- V_{right}$ (current leaving right)

But current leaving right $= I = -$ current leaving left gives.

$$P = I\,(V_{left} - V_{right}) \quad \Rightarrow \quad P = I\,\Delta V \qquad \blacksquare$$

This proof will work for any situation where: