Mon:

Tues: HW

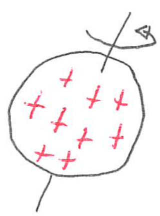
Next Fri: Exam

Phys 311 2020

Lecture 34

Electrodynamics

Electrostatics and magnetostatics consider situations where charge and current densities do not vary with time (although they may vary with location).



Rotating sphere with uniform charge density ρ has current density

$$\vec{J} = \rho \vec{v}$$

Depends on location via $\vec{v}$

Source charges and currents produce electric + magnetic fields that satisfy

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

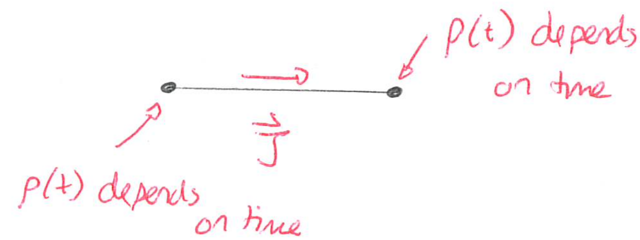
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

These fields are independent of time

Fields computed via:

* Coulomb's Law
* Gauss' Law
* Electrostatic Potential
* Biot-Savart Law
* Ampère's Law
* Magnetic vector potential
* Poisson eqn
* Laplace eqn

However more general situations will have charge and current densities that do vary with time. We will need to see if the scheme for electromagnetic theory so far describes such situations adequately. We have already seen an inconsistency with Ampère's law and a finite current segment.



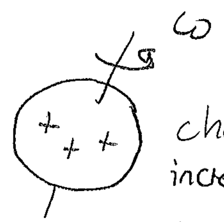
Some examples of these situations are:

1) <u>Time dependent motion of a charge distribution</u>

e.g. sphere whose angular acceleration $\neq 0$

$$\vec{J} = \rho \vec{\omega} \times \vec{r}$$

depends on time.



charges have increasing speed

2) <u>Moving point charge</u>

trajectory : $\vec{s}(t)$

$$\rho(\vec{r},t) = q\,\delta(\vec{r} - \vec{s}(t))$$

3) <u>Alternating current circuits</u>

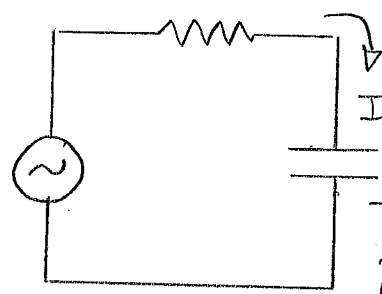
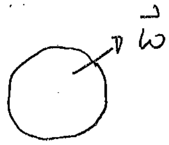


plate charges vary with time.

<u>Continuity equation</u>

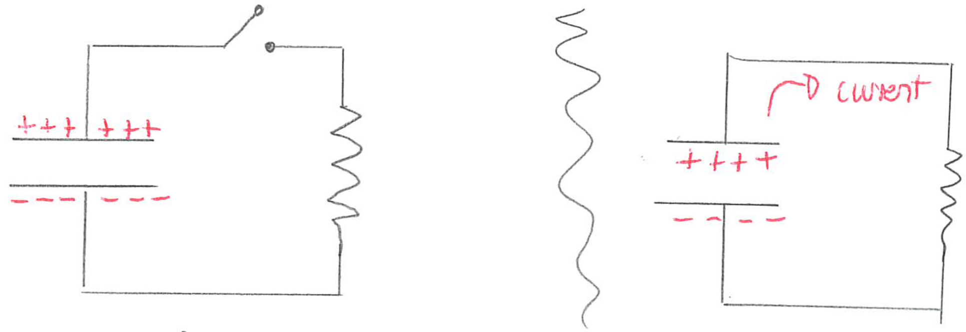
A crucial connection between current and charge distribution arises from the assumption that current only arises from moving point charges. For example for a rotating sphere with charge density ρ

$$\vec{J} = \rho \vec{v} \Rightarrow \vec{J} = \rho \vec{\omega} \times \vec{r}$$



If ρ is constrained to be time independent then $\vec{J}$ must be constrained in some fashion as well.

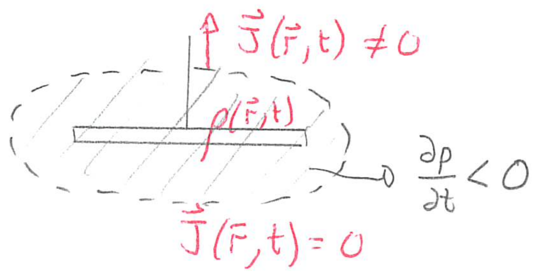
Consider an example of a discharging capacitor. We can describe the situation via a charge density on one plate



Before

During discharge.

The charge density on the upper plate changes with time and this change is related to the current in the wire. Consider a closed region containing the upper plate



Inside this region

$$\frac{\partial \rho}{\partial t} \neq 0$$

This is accompanied by a spatial variation in $\vec{J}$. So we expect

$$\frac{\partial \rho}{\partial t}$$

to be related to $\vec{\nabla} \cdot \vec{J}$.

We can show in general that all charge and current densities satisfy the <u>continuity equation</u>:

$$\boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

1 Continuity equation

Consider any region $\mathcal{R}$ bounded by a closed surface $\mathcal{S}$. The total charge within this region is

$$Q = \int_{\mathcal{R}} \rho(\mathbf{r}', t) \mathrm{d}\tau'.$$

The total current that flows out of the surface is

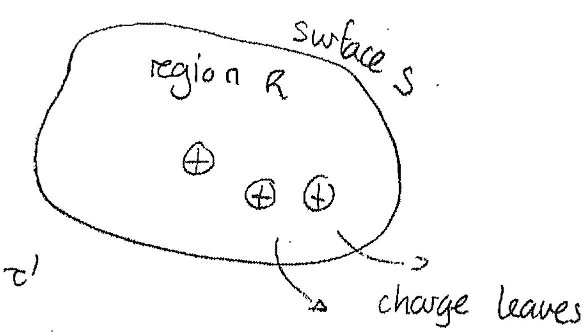
$$I = \oint_{\mathcal{S}} \mathbf{J}(\mathbf{r}', t) \cdot \mathrm{da}.$$

Use the conservation of charge for the region and the divergence theorem to show that

$$\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

<u>Answer:</u> Charge conservation requires

$$I = -\frac{dQ}{dt}$$



$\Rightarrow \dfrac{dQ}{dt} < 0$

$\Rightarrow I > 0$

$$\Rightarrow \quad \oint_S \vec{J}(\vec{r}', t) \cdot d\vec{a} = -\frac{d}{dt} \int_R \rho(\vec{r}', t) d\tau'$$

$$\Rightarrow \quad \oint_S \vec{J}(\vec{r}', t) \cdot d\vec{a} = \int_R -\frac{\partial \rho}{\partial t} d\tau'$$

Then the divergence theorem implies

$$\oint_S \vec{J} \cdot d\vec{a} = \int_R \vec{\nabla} \cdot \vec{J} \, d\tau'$$

Thus

$$\int_R \vec{\nabla} \cdot \vec{J} \, d\tau' = \int_R -\frac{\partial \rho}{\partial t} d\tau' \quad \Rightarrow \quad \int_R \left[\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right] d\tau' = 0$$

This must be true regardless of the region. Thus

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

2 Continuity equation for spherical distributions.

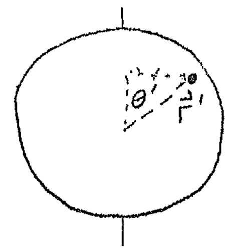
Various solid spheres have radius R.

a) Suppose that a sphere has a uniform charge density ρ and rotates with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$. Determine the current density and show that the continuity equation is valid.

b) At $t = 0$ a sphere has charge density $\rho = \alpha\cos\phi$ and this is fixed to the sphere. The sphere rotates with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$. Show that the charge and current densities are time dependent. Verify that they satisfy the continuity equation.

<u>Answer:</u> a) $\vec{J} = \rho\vec{v}$

$\vec{v} = \omega r \sin\theta\, \hat{\phi}$



$\Rightarrow \vec{J} = \rho\omega r \sin\theta\, \hat{\phi}$

Then $\frac{\partial \rho}{\partial t} = 0$. Now

$$\vec{\nabla}\cdot\vec{J} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \underbrace{J_r}_{0}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \underbrace{J_\theta}_{0}\right) + \frac{1}{r\sin\theta}\frac{\partial J_\phi}{\partial\phi}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\rho\omega r\cos\theta\right) = 0$$

So $\vec{\nabla}\cdot\vec{J} = 0$

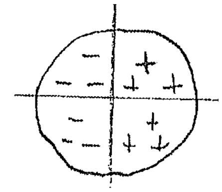
$\frac{\partial\rho}{\partial t} = 0$

and thus $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$.

b) Again

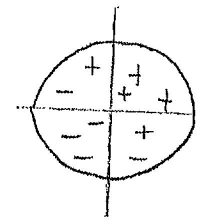
$$\vec{J} = \rho \vec{v}$$
$$= \rho \omega r \sin\theta \, \hat{\phi}$$

t=0



However ρ depends on time. Then

t later



$$\rho(r,\theta,\phi,t) = \alpha \cos(\phi - \omega t)$$

Thus

$$\vec{J} = \alpha \omega r \cos(\phi - \omega t) \sin\theta \, \hat{\phi}$$

Then

$$\frac{\partial \rho}{\partial t} = \alpha \omega \sin(\phi - \omega t)$$

and

$$\vec{\nabla} \cdot \vec{J} = \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} \left[\alpha \omega r \cos(\phi - \omega t) \sin\theta \right)$$

$$= -\frac{1}{\cancel{r}\cancel{\sin\theta}} \alpha \omega \cancel{r} \sin(\phi - \omega t) \cancel{\sin\theta}$$

So $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}$

$$= -\alpha \omega \sin(\phi - \omega t) + \alpha \omega \sin(\phi - \omega t)$$

$$= 0 \qquad \blacksquare$$