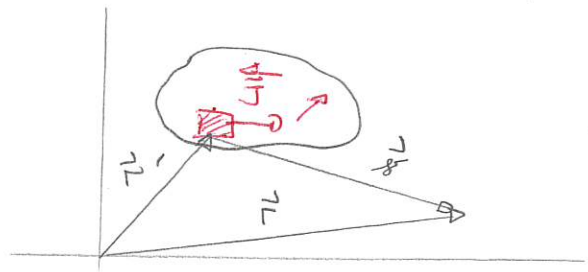
Fri: HW
Read 7.1.1

Magnetic Multipole Expansion

For a localized current distribution the magnetic vector potential can be expressed as a series:



$$\vec{A}(\vec{r}) = \vec{A}_{mon}(\vec{r}) + \vec{A}_{dip}(\vec{r}) + \vec{A}_{quad}(\vec{r}) + \dots$$

Here the monopole term is

$$\vec{A}_{mon}(\vec{r}) = 0$$

The dipole term, often the most significant, is:

$$\boxed{\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2} \int_{\text{all space}} (\hat{r} \cdot \vec{r}\,') \, \vec{J}(\vec{r}\,') \, d\tau'}$$

where the vectors are as illustrated. We now want to separate this into contributions derived entirely from the charge distribution (primed) and those pertaining to the field point. Consider first one dimensional loops.

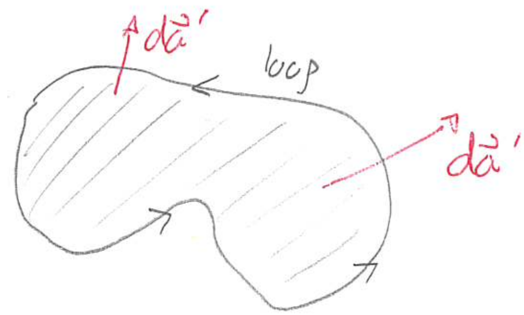
For a one dimensional loop $\vec{J}(\vec{r}')d\tau' \rightarrow I\,d\vec{l}'$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} I \int (\hat{r}\cdot\vec{r}')\,d\vec{l}'$$

We can simplify this using a theorem:

For any constant vector $\vec{c}$ and any loop

$$\oint_{loop} (\vec{c}\cdot\vec{r}')\,d\vec{l}' = \int_{surface} d\vec{a}' \times \vec{c}$$



where the surface is any surface that is banded by the loop.

In the case of

$$\oint (\hat{r}\cdot\vec{r}')\,d\vec{l}'$$

we set $\vec{c} \equiv \hat{r}$ and get

$$\oint_{loop} (\hat{r}\cdot\vec{r}')\,d\vec{l}' = \left[\int_{surface} d\vec{a}'\right] \times \hat{r}$$

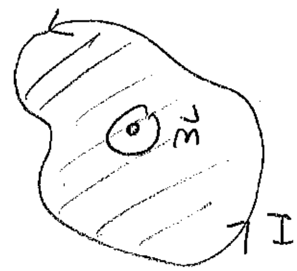
Thus

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2} \underbrace{\left[I \int_{\substack{loop\\surface}} d\vec{a}'\right]}_{} \times \hat{r}$$

Only depends on current configuration.

Thus we define:

Suppose a one dimensional current I flows around a closed loop. Then the magnetic dipole moment is



$$\vec{m} = \int_{\text{loop surface}} I \, d\vec{a}'$$

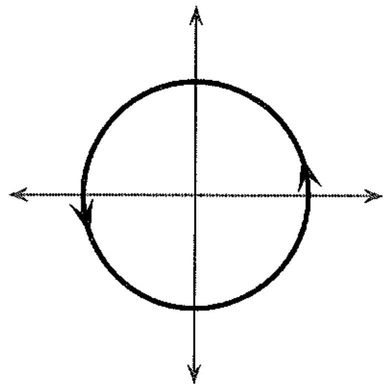
where the integral is over any surface that bounds the current loop. The direction of $d\vec{a}'$ is determined via the r.h. rule and the current direction.

It then follows that at location $\vec{r}$

$$\boxed{\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r} = \frac{\mu_0}{4\pi r^3} \vec{m} \times \vec{r}}$$

1 Magnetic dipole for a circular loop

A circular wire with radius R carries current I. Assume that the loop lies in the xy plane.



a) Determine the magnetic dipole moment of the loop.

b) Determine the dipole magnetic vector potential at all points.

c) Determine the magnetic dipole field at all points.

Answer:

a) $\vec{m} = I \int_{\text{loop}} d\vec{a}'$

Here $d\vec{a}' = da' \hat{z}$

$\Rightarrow \vec{m} = I \int_{\text{surface}} da' \hat{z} \Rightarrow \vec{m} = IA\hat{z}$

b) $\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r}$

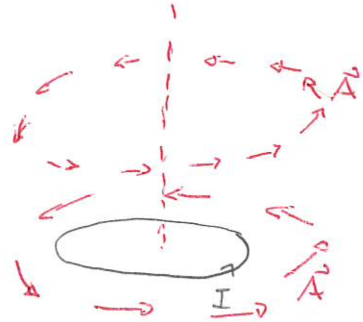
$= \frac{\mu_0}{4\pi r^2} I A \hat{z} \times \hat{r}$

In spherical co-ords

$\hat{z} = \cos\theta \hat{r} - \sin\theta \hat{\theta}$

$\hat{z} \times \hat{r} = -\sin\theta \underbrace{\hat{\theta} \times \hat{r}}_{-\hat{\phi}} = \sin\theta \hat{\phi}$

$$\Rightarrow \quad \vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2} I A \sin\theta \, \hat{\phi}$$



This agrees with what we had obtained in the xy plane previously $(\theta = \pi/2)$

c) $$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \cancelto{0}{\frac{\partial A_\theta}{\partial\phi}}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\cancelto{0}{\frac{\partial A_r}{\partial\phi}} - \frac{\partial}{\partial r}(r A_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r \cancelto{0}{A_\theta}) - \cancelto{0}{\frac{\partial A_r}{\partial\theta}}\right]\hat{\phi}$$

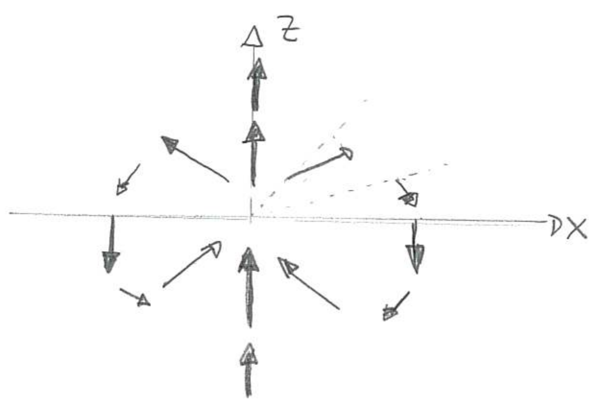
$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\mu_0}{4\pi r^2} IA\sin\theta\right)\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\mu_0}{4\pi r^2} IA\sin\theta\right]\hat{\theta}$$

$$= \frac{1}{r\sin\theta}\frac{\mu_0 IA}{4\pi r^2}\frac{\partial}{\partial\theta}(\sin^2\theta)\,\hat{r} - \frac{1}{r}\frac{\mu_0}{4\pi}\sin\theta\, IA\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\theta}$$

$$= \frac{\mu_0}{4\pi r^3} IA\, 2\cos\theta\,\hat{r} + \frac{\mu_0}{4\pi r^3} IA\sin\theta\,\hat{\theta}$$

$$= \frac{\mu_0}{4\pi r^3} IA\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

Plotting gives



Demo · PSU-S Dipole

General description of monopole and dipole moments

Consider the monopole moment for a general current

$$\vec{A}_{mon} = \frac{\mu_0}{4\pi r} \int_{\text{all space.}} \vec{J}(\vec{r}')\, d\tau'$$

We assume that the current is stationary. So $\vec{\nabla}\cdot\vec{J} = 0$ and $\vec{\nabla}'\cdot\vec{J}(\vec{r}') = 0$

Then the following theorem is useful:

Theorem: For any vector field $\vec{v}$

$$\vec{v} = \vec{\nabla}\cdot(x\vec{v})\,\hat{x} + \vec{\nabla}\cdot(y\vec{v})\,\hat{y} + \vec{\nabla}\cdot(z\vec{v})\,\hat{z} - \vec{r}\,(\vec{\nabla}\cdot\vec{J})$$

Proof: $$\vec{\nabla}\cdot(x\vec{v}) = \underbrace{(\vec{\nabla}x)}_{\hat{x}}\cdot\vec{v} + x\,\vec{\nabla}\cdot\vec{v}$$

$$= \hat{x}\cdot\vec{v} + x\,\vec{\nabla}\cdot\vec{v}$$

$$\Rightarrow \quad \vec{\nabla}\cdot(x\vec{v})\,\hat{x} = (\hat{x}\cdot\vec{v})\,\hat{x} + x\hat{x}\;\vec{\nabla}\cdot\vec{v}$$

Repeating this for the other terms and adding gives:

$$\vec{\nabla}\cdot(x\vec{v})\,\hat{x} + \vec{\nabla}\cdot(y\vec{v})\,\hat{y} + \vec{\nabla}\cdot(z\vec{v})\,\hat{z} = (\hat{x}\cdot\vec{v})\,\hat{x} + (\hat{y}\cdot\vec{v})\,\hat{y} + (\hat{z}\cdot\vec{v})\,\hat{z} + \underbrace{(x\hat{x} + y\hat{y} + z\hat{z})}_{\vec{r}}\,\vec{\nabla}\cdot\vec{v}$$

Then $\vec{v} = (\hat{x}\cdot\vec{v})\,\hat{x} + (\hat{y}\cdot\vec{v})\,\hat{y} + (\hat{z}\cdot\vec{v})\,\hat{z}$ gives the result. $\boxtimes$

So

$$\vec{J}(\vec{r}') = \vec{\nabla}\cdot(x'\vec{J})\,\hat{x} + \vec{\nabla}\cdot(y'\vec{J})\,\hat{y} + \vec{\nabla}\cdot(z'\vec{J})\,\hat{z} - \vec{r}'\,\underbrace{\vec{\nabla}'\cdot\vec{J}}_{=0}$$

and

$$\int \vec{J}(\vec{r}')\,d\tau' = \int_{\text{all space}} \vec{\nabla}\cdot(x'\vec{J})\,d\tau'\,\hat{x} + \cdots$$

$$= \left[\int_{\text{all space}} x'\vec{J}\cdot d\vec{a}\right]\hat{x} + \cdots$$

But the integral must be taken over all space. If the current is localized then $\vec{J}(\vec{r}')=0$ on the boundary of "all space" and thus each integral is zero. So for stationary currents

$$\int_{\text{all space}} \vec{J}(\vec{r}')\,d\tau' = 0$$

and thus:

$$\boxed{\text{For localized stationary currents, } \vec{A}_{mon} = 0}$$

Dipole term:

The dipole term

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2}\int_{\text{all space}} \vec{J}(\vec{r}')(\hat{r}\cdot\vec{r}')\,d\tau' = \frac{\mu_0}{4\pi r^3}\int J(\vec{r}')(\vec{r}\cdot\vec{r}')\,d\tau'$$

involves the integral

$$\int \vec{J}(\vec{r}')\,\vec{r}\cdot\vec{r}'\,d\tau'$$

involves

$$\vec{r}\cdot\vec{r}' = xx' + yy' + zz'$$

and becomes

$$\int \vec{J}(\vec{r}')\,\vec{r}\cdot\vec{r}'\,d\tau' = \left[\int x'\,\vec{J}(\vec{r}')\,d\tau'\right] x$$

$$+ \left[\int y'\,\vec{J}(\vec{r}')\,d\tau'\right] y$$

$$+ \left[\int z'\,\vec{J}(\vec{r}')\,d\tau'\right] z$$

We will use the following identities:

Theorem: If $\vec{\nabla}' \cdot \vec{J}(r') = 0$ then

1) $\vec{\nabla}' \cdot [x'^2 \vec{J}(r')] = 2x' J_x$

2) $\vec{\nabla}' \cdot [x'y' \vec{J}(\vec{r}')] = (\vec{r}' \times \vec{J}) \cdot \hat{z} + 2y' J_x$

$\vec{\nabla}' \cdot [y'z' \vec{J}(\vec{r}')] = (\vec{r}' \times \vec{J}) \cdot \hat{x} + 2z' J_y$

$\vec{\nabla}' \cdot [x'z' \vec{J}(\vec{r}')] = (\vec{r}' \times \vec{J}) \cdot \hat{y} + 2x' J_z$

$\vec{\nabla}' \cdot [x'y' \vec{J}(\vec{r}')] = -(\vec{r}' \times \vec{J}) \cdot \hat{z} + 2x' J_y$

$\vec{\nabla}' \cdot [y'z' \vec{J}(\vec{r}')] = -(\vec{r}' \times \vec{J}) \cdot \hat{x} + 2y' J_z$ etc..

Proof: 1) $\vec{\nabla}' \cdot (x'^2 \vec{J}) = (\vec{\nabla} x'^2) \cdot \vec{J} + x'^2 \cancel{\vec{\nabla}' \cdot \vec{J}(\vec{r}')} = 2x' J_x$ ✓

2) $\vec{\nabla}' \cdot [x'y' \vec{J}] = \vec{\nabla}'(x'y') \cdot \vec{J} + x'y' \cancel{\vec{\nabla}' \cdot \vec{J}(\vec{r}')}$

$= (y' \hat{x} + x' \hat{y}) \cdot \vec{J}$

$= y' J_x + x' J_y$

$= x' J_y - y' J_x + 2y' J_x$

$= [\vec{r}' \times \vec{J}(\vec{r}')] \cdot \hat{z} + 2y' J_x$ ✓

$\vec{\nabla} \cdot (x'y' \vec{J}) = y' J_x + x' J_y = y' J_x - x' J_y + 2x' J_y$

$= -(\vec{r}' \times \vec{J}) + 2x' J_y$ ✓

We now use these to show that

If $\vec{\nabla}' \cdot \vec{J}(\vec{r}\,') = 0$ and the current is localized then

$$\int x' \vec{J}(\vec{r}\,')\, d\tau' = m_z \hat{y} - m_y \hat{z}$$

where

$$\vec{m} = \frac{1}{2}\int \vec{r}\,' \times \vec{J}(\vec{r}\,')\, d\tau'$$

To show this

$$\int x' \vec{J}(\vec{r}\,') d\tau' = \left[\int x' J_x\, d\tau'\right] \hat{x} + \left[\int x' J_y\, d\tau'\right] \hat{y} + \dots$$

and

$$\int_{\text{all space}} x' J_x(\vec{r}\,') d\tau' = \frac{1}{2}\int_{\text{all space}} \vec{\nabla}' \cdot (x'^2 \vec{J})\, d\tau' = \frac{1}{2}\int_{\text{all space}} x'^2 \vec{J} \cdot d\vec{a}\,'$$

and on the boundary of all space $\vec{J} = 0 \Rightarrow \int_{\text{all space.}} x' J_x\, d\tau' = 0$

Then

$$\int x' J_y\, d\tau' = \underbrace{\frac{1}{2}\int \vec{\nabla} \cdot \left[x' y' \vec{J}(\vec{r}\,')\right] d\tau'}_{=0 \text{ by same argument}} + \frac{1}{2}\int (\vec{r}\,' \times \vec{J}(\vec{r}\,')) \cdot \hat{x}\, d\tau'$$

So $$\int x' J_y(\vec{r}\,') d\tau' = \frac{1}{2}\int \left[\vec{r}\,' \times \vec{J}(\vec{r}\,')\right] \cdot \hat{z}\, d\tau' = m_z$$

by the definition of $\vec{m}$. Similarly

$$\int x' J_z(\vec{r}\,') d\tau' = - m_y$$

Thus

$$\int x\, \vec{J}(\vec{r}')d\tau' = m_z\hat{y} - m_y\hat{z}$$

Similarly

$$\int y'\, \vec{J}(\vec{r}')d\tau' = m_x\hat{z} - m_z\hat{x}$$

$$\int z'\, \vec{J}(\vec{r}')d\tau' = m_y\hat{x} - m_x\hat{y}$$

So $$\int \vec{J}(\vec{r}')\, \vec{r}\cdot\vec{r}'\, d\tau' = xm_z\hat{y} - xm_y\hat{z}$$
$$+ ym_x\hat{z} - ym_z\hat{x}$$
$$+ zm_y\hat{x} - zm_x\hat{y}$$
$$= (zm_y - ym_z)\hat{x}$$
$$+ (xm_z - zm_x)\hat{y}$$
$$+ (ym_x - xm_y)\hat{z}$$
$$= \vec{m}\times\vec{r}$$

$$\Rightarrow \quad \vec{A}_{dip} = \frac{\mu_0}{4\pi r^2}\vec{m}\times\vec{r}$$

So we have.

For a localized stationary current distribution the magnetic dipole moment is

$$\vec{m} = \frac{1}{2}\int_{\text{all space}} \vec{r}'\times\vec{J}(\vec{r}')d\tau'$$

and the magnetic dipole potential is

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi r^3}\vec{m}\times\vec{r}$$

In one dimension

$$\boxed{\vec{m} = \frac{1}{2} \int \vec{r}' \times \vec{I} \, dl'}$$

In two dimensions

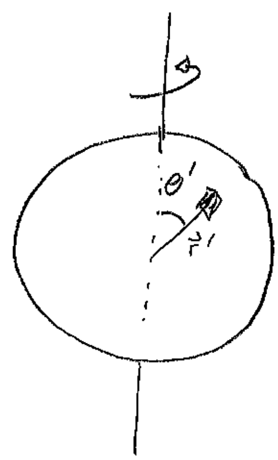
$$\boxed{\vec{m} = \frac{1}{2} \int \vec{r}' \times \vec{K} \, da'}$$

2 Magnetic dipole moment of a sphere

A solid sphere with radius R has a uniform volume charge density. The sphere rotates with constant angular velocity ω. Determine the magnetic dipole moment of the sphere in terms of the volume charge density and the total charge.

Answer:

$$\vec{m} = \frac{1}{2}\int_{\text{sphere}} \vec{r}\,' \times \vec{J}(\vec{r}\,')\, d\tau'$$



We need the current density.

$$\vec{J}(\vec{r}\,') = \rho\, \vec{v}(\vec{r}\,')$$

Then at $\vec{r}\,'$ the velocity is

$$\vec{v}(\vec{r}\,') = \omega r' \sin\theta' \hat{\phi}$$

Now $\vec{r}\,' = r'\hat{r}$ gives

$$\vec{r}\,' \times \vec{J}(\vec{r}\,') = \rho r' \omega r' \sin\theta' \underbrace{\hat{r} \times \hat{\phi}}_{-\hat{\theta}}$$

$$= \rho \omega r'^2 \sin\theta' (-\hat{\theta})$$

$$= -\rho \omega r'^2 \sin\theta' \hat{\theta}$$

Again $\hat{\theta}$ varies and so

$$\hat{\theta} = \cos\theta' \cos\phi' \hat{x} + \cos\theta' \sin\phi' \hat{y} - \sin\theta' \hat{z}$$

$$\Rightarrow \vec{r}\,' \times \vec{J}(\vec{r}\,') = -\rho\omega r'^2 \left[\cos\theta' \cos\phi' \hat{x} + \cos\theta' \sin\phi' \hat{y} - \sin\theta' \hat{z}\right] \sin\theta'$$

So

$$\vec{m} = \int_0^R dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \, r'^2 \sin\theta' \, \vec{r}' \times \vec{J}(\vec{r}')$$

We can see that the integrals w.r.t. ϕ' give 0 for $\hat{x}, \hat{y}$ terms.

Thus:

$$\vec{m} = 2\pi \rho \omega \underbrace{\int_0^R dr' \, r'^4}_{\frac{R^5}{5}} \underbrace{\int_0^{2\pi} d\theta' \sin^3\theta'}_{4/3} \hat{z}$$

$$\vec{m} = \frac{8\pi \rho \omega R^5}{15} \hat{z}$$

$$\Rightarrow \vec{m} = \frac{8\pi\rho}{15} R^5 \vec{\omega} \qquad \text{where } \vec{\omega} = \omega \hat{z}$$

Then the charge density is

$$\rho = \frac{Q}{4\pi R^3/3} = \frac{3Q}{4\pi R^3}$$

Thus

$$\vec{m} = \frac{2}{5} Q R^2 \vec{\omega}$$

Note that the total angular momentum is $\vec{L} = I\vec{\omega}$ and thus

$$\vec{m} = \frac{\frac{2}{5} Q R^2}{I} \vec{\omega}$$

For a uniform mass distribution, $I = \frac{2}{5} M R^2$

$$\Rightarrow \vec{m} = \frac{Q}{M} \vec{L}$$