Phys 311 2020

Weds: Read 5.4.3.

Lecture 32

Magnetic Vector Potential

We showed that, for any magnetic field $\vec{B}$:

> There exists a vector potential $\vec{A}$ such that
>
> 1) $\vec{B} = \vec{\nabla} \times \vec{A}$
>
> 2) $\vec{\nabla} \cdot \vec{A} = 0$

Then the vector potential is related to the source current $\vec{J}$ via:

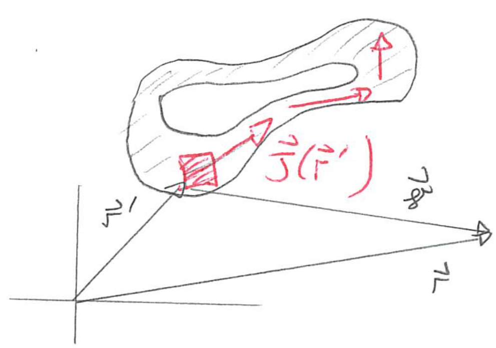
$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

This is essentially three independent Poisson equations, one for each component. If the source current is localized then the equations can be solved as the scalar Poisson eqn:

$$\nabla^2 V = \rho/\epsilon_0 \quad \Rightarrow \quad V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{\mathscr{r}} d\tau'$$

We then replace $\epsilon_0 \to \frac{1}{\mu_0}$ $\quad \rho \to J_x$ etc,... This gives:

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J(\vec{r}\,')}{\mathscr{r}} d\tau'}$$



We can use this to determine the field due to spherically symmetric rotating charge distributions

Example: A sphere with radius R contains a spherically symmetric charge distribute

$$\rho = \rho(r')$$

The sphere rotates with a constant angular velocity with magnitude ω. Determine

a) $\vec{A}$ everywhere

b) $\vec{B}$ everywhere

Answer: Suppose that the sphere's angular velocity is $\vec{\omega}$ and we want the field at some point $\vec{r}$. Then

$$\vec{J}(\vec{r}') = \rho \vec{v} = \rho\, \vec{\omega} \times \vec{r}'$$

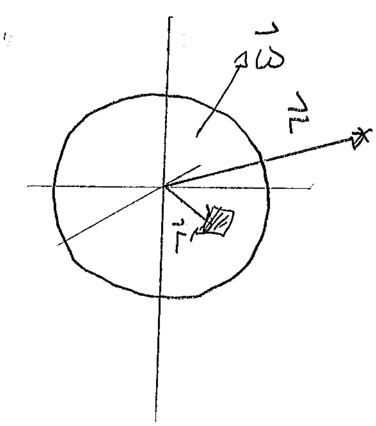
and

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{\mathscr{r}}\, d\tau'$$

We need to determine $\vec{r}, \vec{r}'$, etc... As usual

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$\Rightarrow \quad \mathscr{r} = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}$$



$$d\tau' = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$$

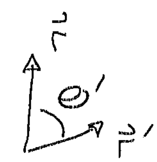
Thus:

$$\vec{A} = \frac{\mu_0}{4\pi} \int_0^R dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \; r'^2 \sin\theta' \; \frac{\rho(r')}{\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}} \; \vec{\omega} \times \vec{r}'$$

$$= \frac{\mu_0}{4\pi} \vec{\omega} \times \left[\int_0^R dr' \, r'^2 \rho(r') \int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\phi' \; \frac{\vec{r}'}{(r^2 + r'^2 - 2\vec{r}\cdot\vec{r}')^{1/2}} \right]$$

We need to evaluate the term with the brackets. To do so we will choose axes so that $\hat{z}$ is along $\vec{r}$. Then

$$\vec{r}\cdot\vec{r}' = r\cos\theta'$$



and so the term in brackets is:

$$\int_0^R dr' \, r'^2 \rho(r') \int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\phi' \; \frac{r' \left(\cos\phi' \sin\theta' \hat{x} + \sin\phi' \sin\theta' \hat{y} + \cos\theta' \hat{z} \right)}{\left[r^2 + r'^2 - 2rr\cos\theta' \right]^{1/2}}$$

The integrals over ϕ' are easily done and leave

$$2\pi \int_0^R dr' \, r'^3 \rho(r') \int_0^\pi d\theta' \sin\theta' \; \frac{\cos\theta'}{\left[r^2 + r'^2 - 2rr'\cos\theta' \right]^{1/2}} \; \hat{z}$$

Now let $u = \cos\theta'$. Then the θ' integral becomes

$$\int_{-1}^{1} \frac{u}{\left[r^2 + r'^2 - 2rr'u \right]^{1/2}} du = - \frac{r'^2 + r^2 + rr'u}{3r^2 r'^2} \left(r^2 + r'^2 - 2rr'u \right)^{1/2} \Big|_{-1}^{1}$$

$$= - \frac{r'^2 + r^2 + rr'}{3r^2 r'} |r - r'| + \frac{r'^2 + r^2 - rr'}{3r^2 r'^2} (r + r')$$

The remaining cases depend on whether $r > r'$ or $r < r'$. If $r > r'$

$$\int \cdots \, du = \frac{1}{3r^2r'^2}\Big[r\big(\cancel{r'^2}+\cancel{r^2}-rr'-\cancel{r'^2}-\cancel{r^2}-rr'\big) + r'\big(r'^2+r^2-\cancel{rr'}+r'^2+r^2+\cancel{rr'}\big)\Big]$$

$$= \frac{1}{3r^2r'^2}\Big[\cancel{-2r^2r'} + \cancel{2r^2r'} + 2r'^3 \Big] = \frac{2r'}{3\,r^2}$$

If $r < r'$

$$\int \cdots \, du = \frac{1}{3r^2r'^2}\Big[r\big(r'^2+r^2-\cancel{rr'}+r'^2+r^2+\cancel{rr'}\big)$$
$$+ r'\big(\cancel{r'^2}+\cancel{r^2}-rr'-\cancel{r'^2}-\cancel{r^2}-rr'\big)\Big]$$

$$= \frac{1}{3r^2r'^2}\Big[2r^3 + \cancel{2rr'^2} - \cancel{2rr'^2}\Big] = \frac{2}{3}\,\frac{r}{r'^2}.$$

So we get that the term in brackets is:

$$2\pi \int_0^R r'^3 \rho(r') \, \frac{2r'}{3r^2} \, dr' \, \hat{z} \qquad \text{if} \quad r > R$$

$$2\pi \left\{ \int_0^r r'^3 \rho(r') \, \frac{2r'}{3r^2} \, dr' + \int_r^R r'^3 \rho(r') \, \frac{2r}{r'^2} \, dr' \right\} \hat{z} \qquad \text{if} \quad r < R$$

Each is proportional to $\hat{z}$. But note $\vec{r} = r\hat{z}$

$$\Rightarrow \hat{z} = \frac{\vec{r}}{r}$$

This can be extracted from the integral.

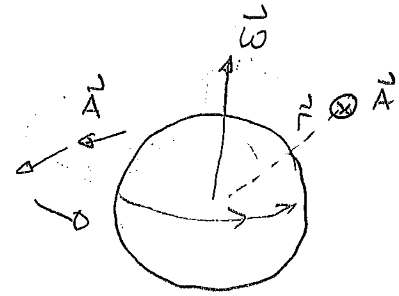
Thus

$$\vec{A} = \frac{\mu_0}{3}\frac{\vec{\omega}\times\vec{r}}{r^3}\left\{\int_0^R r'^4 \rho(r')\,dr'\right\} \qquad \text{if } r>R \quad \text{outside.}$$

$$\vec{A} = \mu_0 \frac{\vec{\omega}\times\vec{r}}{r}\left\{\int_0^r \frac{1}{3}\frac{r'^4}{r^2}\rho(r')\,dr' + r\int_r^R r'\rho(r')\,dr'\right\} \qquad \text{if } r<R \quad \text{inside.}$$

We can see that the basic form of the vector potential depends on the charge distribution. Outside of the sphere it is dependent, not only on the total charge, but also how that total charge is distributed. In all cases the magnetic vector potential circles $\vec{\omega}$

Suppose that



a) charge is uniform:

$$\rho = \frac{Q}{4/3\,\pi R^3}$$

Then:

$$\vec{A} = \frac{\mu_0}{3}\frac{\vec{\omega}\times\vec{r}}{r^3}\left[\rho\frac{R^5}{5}\right] = \frac{\mu_0}{3}\frac{\vec{\omega}\times\vec{r}}{r^3}\,Q\,\frac{3R^2}{20\pi} \qquad \text{outside}$$

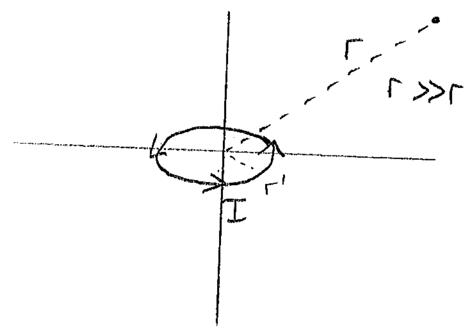
b) charge is distributed as $\rho(r') = \alpha r'$ $\Rightarrow$ $Q = 4\pi\alpha R^4/4 = \pi\alpha R^4$.

$$\vec{A} = \frac{\mu_0}{3}\frac{\vec{\omega}\times\vec{r}}{r^3}\left[\alpha\frac{R^6}{6}\right] = \frac{\mu_0}{3}\frac{\vec{\omega}\times\vec{r}}{r^3}\,Q\left[\frac{R^2}{6\pi}\right] \qquad \text{outside.}$$

These are distinct...

Magnetic multipole expansion

Consider situations where we cannot determine the magnetic vector potential directly, but the current distribution is localized. Can we approximate $\vec{A}$ at large distances? We will show that this is possible via a multipole expansion. Then



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

and we need to approximate $\mathscr{r} = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}}$. Here

$$\vec{\mathscr{r}}\cdot\vec{\mathscr{r}} = (\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')$$

$$= r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'$$

So

$$\mathscr{r} = \left[r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'\right]^{1/2} = r\left[1 - 2\frac{\vec{r}\cdot\vec{r}\,'}{r} + \left(\frac{r'}{r}\right)^2\right]^{1/2}$$

Then a Taylor series gives:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3}x^3 + \dots$$

and with $x = -\frac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \left(\frac{r'}{r}\right)^2$ and $n = -1/2$. So

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'^2}{r^2} - 2\frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right) + \frac{3}{8}\left(\frac{r'^2}{r^2} - 2\frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right)^2 + \dots\right]$$

and

$$\frac{\vec{r}\cdot\vec{r}\,'}{r^2} = \frac{\vec{r}}{r}\cdot\frac{\vec{r}\,'}{r} = \frac{\hat{r}\cdot\vec{r}\,'}{r^2}$$

So

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left\{ 1 + \frac{\hat{r}\cdot\vec{r}\,'}{r} - \frac{r'^2}{2r^2} + \frac{3}{8}\left[\frac{r'^4}{r^4} + 4\left(\frac{\hat{r}\cdot\vec{r}\,'}{r}\right)^2 - 4\,\frac{\hat{r}\cdot\vec{r}\,'}{r^3}\,r'^2 \right] + \dots \right\}$$

$$= \frac{1}{r}\left\{ 1 + \underbrace{\frac{\hat{r}\cdot\vec{r}\,'}{r}}_{\text{order } r'/r} + \frac{1}{2}\underbrace{\left[3\left(\frac{\hat{r}\cdot\vec{r}\,'}{r}\right)^2 - \left(\frac{r'}{r}\right)^2 \right]}_{\text{order } (r'/r)^2} + \dots \right\}$$

Thus:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi r}\int_{\text{all space}} J(\vec{r}\,')\left\{ 1 + \frac{\hat{r}\cdot\vec{r}\,'}{r} + \frac{1}{2}\left[3\left(\frac{\hat{r}\cdot\vec{r}\,'}{r}\right)^2 - \frac{r'^2}{r^2} \right] + \dots \right\} d\tau'$$

$$= \frac{\mu_0}{4\pi r}\int_{\text{all space}} \vec{J}(\vec{r}\,')\,d\tau' \quad \longleftarrow \text{order } 1/r$$

$$+ \frac{\mu_0}{4\pi r^2}\int_{\text{all space}} (\hat{r}\cdot\vec{r}\,')\,\vec{J}(\vec{r}\,')\,d\tau' \quad \longleftarrow \text{order } 1/r^2$$

$$+ \frac{\mu_0}{4\pi r^3}\int_{\text{all space}} \frac{1}{2}\left[3(\hat{r}\cdot\vec{r}\,') - r'^2 \right] \vec{J}(\vec{r}\,')\,d\tau' \quad \longleftarrow \text{order } 1/r^3$$

$$+ \dots$$

We can thus rewrite this as a succession of terms:

$$\vec{A}(\vec{r}) = \vec{A}_{mon}(\vec{r}) + \vec{A}_{dip}(\vec{r}) + \vec{A}_{quad}(\vec{r}) + \dots$$

The terms are:

Monopole term:

$$\boxed{\vec{A}_{mon} = \frac{\mu_0}{4\pi r} \int_{\text{all space}} \vec{J}(\vec{r}')d\tau'}$$

Dipole term:

$$\boxed{\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} \int_{\text{all space}} (\hat{r}\cdot\vec{r}')\, \vec{J}(\vec{r}')d\tau'}$$

Quadrupole term

$$\boxed{\vec{A}_{quad} = \frac{\mu_0}{4\pi r^3} \int_{\text{all space}} \vec{J}(\vec{r}')\, \frac{1}{2}\left[3(\hat{r}\cdot\vec{r}') - r'^2\right] d\tau'}$$

Magnetic monopole term

For any localized current distribution,

$$\int \vec{J}(\vec{r}')\, d\tau' = 0$$

and thus:

$$\vec{A}_{mon} = 0$$

for localized currents. So for localized currents there are no monopole terms.

Magnetic dipoles

The dipole term is

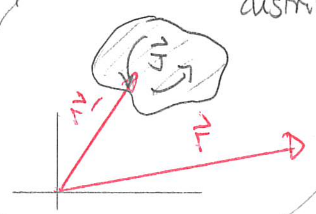
$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\int (\hat{r}\cdot\vec{r}\,')\,\vec{J}(\vec{r}\,')\,d\tau'$$

field location

refers to charge distrib.

We eventually aim to separate contributions from the charge distribution from those of the field point. Eventually we will get

current distrib.



→ Determine a magnetic dipole moment $\vec{m}$ using only $\vec{J}(\vec{r}\,')$ and $\vec{r}\,'$

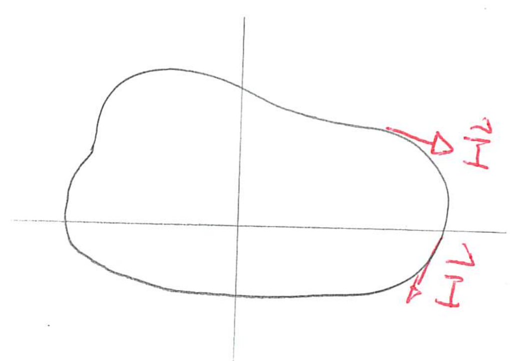
This gives a co-ordinate independent vector, $\vec{m}$

Compute dipole potential using $\vec{m}$ and field vector $\vec{r}$ only

Consider line currents. Then

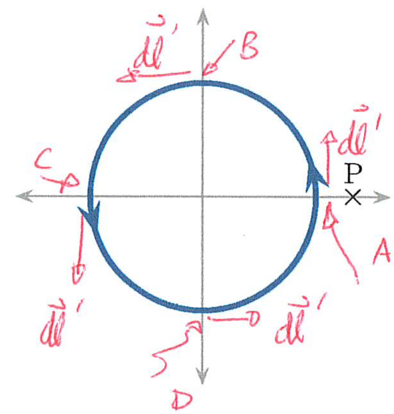
$$\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2}\int (\hat{r}\cdot\vec{r}\,')\,\vec{I}\,dl'$$

$$= \frac{\mu_0}{4\pi r^2}\, I \underbrace{\int_{loop} (\hat{r}\cdot\vec{r}\,')\,d\vec{l}\,'}_{\text{only refers to loop configuration!}}$$



1 Magnetic dipole for a circular loop

A circular wire with radius R carries current I. The goal of this exercise is to determine the dipole magnetic vector potential at point P at a distance r from the origin.



a) Determine the direction of the contribution to the magnetic vector potential at P from the four points where ring intersects the axes.

b) Evaluate the dipole magnetic vector potential at P produced by the entire loop.

c) Express the result in terms of the area of the loop.

d) What would the dipole magnetic vector potential be at any location in the plane?

Answers: a) The integrand contributes

$(\hat{r}\cdot\vec{r}\,')\,\vec{dl}'$ and $\hat{r} = \hat{x}$

Point	Vectors		Contributes
A	$\hat{r}\cdot\vec{r}\,' = R$	$\vec{dl}'\ \uparrow$	$R\,dl'\ \uparrow$
B	$\hat{r}\cdot\vec{r}\,' = 0$	$\vec{dl}'\ \leftarrow$	0
C	$\hat{r}\cdot\vec{r}\,' = -R$	$\vec{dl}'\ \downarrow$	$R\,dl'\ \uparrow$
D	$\hat{r}\cdot\vec{r}\,' = 0$	$\vec{dl}'\ \rightarrow$	0

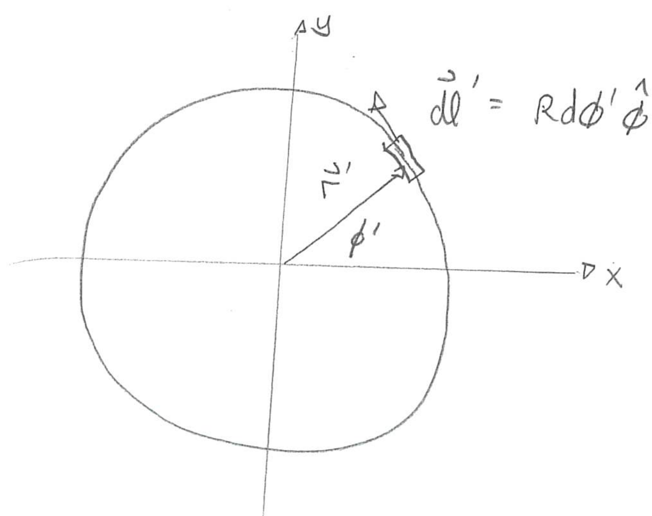
b) Consider any point on the loop

$\vec{r}\,' = R\cos\phi'\,\hat{x} + R\sin\phi'\,\hat{y}$

$\hat{r} = \hat{x}$

$\Rightarrow \hat{r}\cdot\vec{r}\,' = R\cos\phi'$

$(\hat{r}\cdot\vec{r}\,')\,\vec{dl}' = R\cos\phi'\,R\,d\phi'\,\hat{\phi}$



So

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} I \int_0^{2\pi} R^2 \cos\phi' d\phi' \ \hat{\phi}$$

But $\hat{\phi} = -\sin\phi' \hat{x} + \cos\phi' \hat{y}$ and

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} I R^2 \int_0^{2\pi} \Big[\underbrace{-\sin\phi' \cos\phi' \hat{x}}_{\text{integrates to zero}} + \cos^2\phi' \hat{y} \Big] d\phi'$$

$$= \frac{\mu_0}{4\pi r^2} I R^2 \underbrace{\int_0^{2\pi} \cos^2\phi' d\phi'}_{\pi} \ \hat{y}$$

$$\Rightarrow \boxed{\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} I \pi R^2 \hat{y}}$$

c) loop area is $A = \pi R^2 \Rightarrow \vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} I A \hat{y}$

d) By symmetry $\vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} I A \hat{\phi}$