Mon: Read 5.4.1, 5.4.3

Tues:

Magnetic Vector Potential

The Biot-Savart Law eventually leads to the constraint on magnetic fields: $\vec{\nabla}\cdot\vec{B} = 0$. This means that:

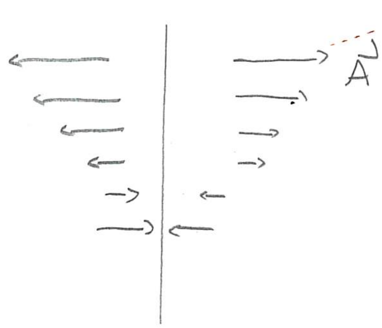
> A magnetic vector potential $\vec{A}$, such that
> $$\vec{B} = \vec{\nabla}\times\vec{A}$$
> could exist

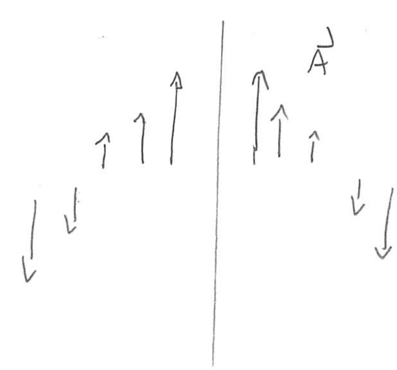
We have seen that, for the example of a field produced by an infinite straight current, i.e.

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi}$$

there are many possible vector potentials. For example:

$$\vec{A} = \left[\frac{\mu_0 I}{2\pi s} z + \underbrace{f(s)}_{\text{any function}}\right]\hat{s} \qquad \vec{A} = \left[-\frac{\mu_0 I}{2\pi}\ln(s) + \underbrace{g(z)}_{\text{any function}}\right]\hat{z}$$





Each of these gives $\vec{\nabla}\times\vec{A} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi}$

Does the vector potential exist in general? We show:

> For any magnetic field $\vec{B}$, there exists a vector potential $\vec{A}$ so that
> $$\vec{B} = \vec{\nabla} \times \vec{A}$$

<u>Proof:</u> Consider Cartesian co-ordinates. Then

$$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

Then $\vec{\nabla} \times \vec{A} = B$ implies

$$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$$\Rightarrow \quad \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x$$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = B_y$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z$$

These can be solved. Note that in general, two components of $\vec{A}$ cannot be zero, since this implies one component of $\vec{B}$ is zero. Thus we seek solutions where one component of $\vec{A}$ is zero. Suppose $A_x = 0$. Then

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x$$

$$-\frac{\partial A_z}{\partial x} = B_y$$

$$\frac{\partial A_y}{\partial x} = B_z$$

Integrating the last two gives:

$$A_z = -\int B_y(x,y,z)\,dx + F(y,z)$$

where F is an arbitrary function of two variables. Likewise

$$A_y = \int B_z(x,y,z)\,dx + G(y,z)$$

where G is another function. Then

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x$$

$$\Rightarrow -\int \frac{\partial B_y}{\partial y}\,dx + \frac{\partial F}{\partial y} - \left\{ \int \frac{\partial B_z}{\partial z}\,dx + \frac{\partial G}{\partial z} \right\} = B_x$$

$$\Rightarrow -\int \underbrace{\left[\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right]}_{= \vec{\nabla}\cdot\vec{B} - \frac{\partial B_x}{\partial x}} dx + \frac{\partial F}{\partial y} - \frac{\partial G}{\partial z} = B_x$$

$$\Rightarrow \underbrace{\int \frac{\partial B_x}{\partial x}\,dx}_{B_x} + \frac{\partial F}{\partial y} - \frac{\partial G}{\partial z} = B_x$$

Thus all we need is $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial z}$. There are many choices, e.g. $F = G = 0$, or $F = F(z)$, $G = G(y)$. This gives:

$$\boxed{\begin{aligned} A_x &= 0 \\ A_y &= -\int B_y\,dx + F(y,z) \qquad \text{with} \quad \frac{\partial F}{\partial y} = \frac{\partial G}{\partial z} \\ A_z &= +\int B_z\,dx + G(y,z) \end{aligned}}$$

1 Vector potential for a solenoid

A solenoid with n turns per unit length carries current I. The solenoid is oriented with its axis along the z axis. The field produced by the solenoid is, in cylindrical coordinates,

$$\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}} & \text{inside} \\ 0 & \text{outside.} \end{cases}$$

a) Using Cartesian coordinates, find differential equations that the vector potential must satisfy inside the solenoid.

b) Assume that $A_x = A_z = 0$. Find and sketch the simplest form of the vector potential inside the solenoid.

c) Assume that $A_y = A_z = 0$. Find and sketch the simplest form of the vector potential inside the solenoid.

d) Using cylindrical coordinates, find differential equations that the vector potential must satisfy inside the solenoid.

e) Assume that $A_s = A_z = 0$. Find and sketch the simplest form of the vector potential inside the solenoid.

Answer: a) $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\Rightarrow \quad B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \mu_0 n I$$

b) The equations give

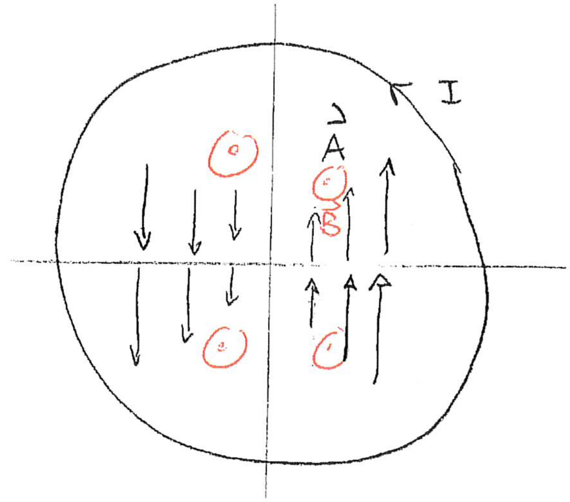
$$\frac{\partial A_y}{\partial z} = 0 \quad \Rightarrow \quad A_y = A_y(x, y)$$

$$\frac{\partial A_y}{\partial x} = \mu_0 n I \quad \Rightarrow \quad A_y = \mu_0 n I x + f(y)$$

The simplest case is $A_y = \mu_0 n I x$

$$\Rightarrow \quad \vec{A} = \mu_0 n I x\, \hat{y}$$

Viewed along the axis



c) In this case:

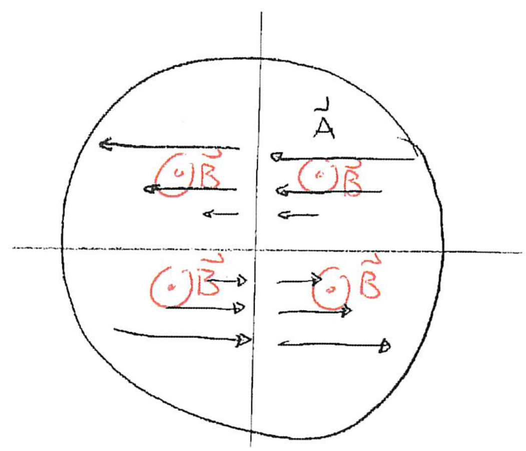
$$\frac{\partial A_x}{\partial z} = 0 \quad \Rightarrow \quad A_x = A_x(x,y)$$

$$-\frac{\partial A_x}{\partial y} = \mu_0 n I \quad \Rightarrow \quad A_x = -\mu_0 n I y + g(y)$$

The simplest solution is

$$\vec{A} = -\mu_0 n I y \hat{x}$$

and this is sketched



d) Here

$$\vec{\nabla} \times \vec{A} = \left[\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s A_\phi) - \frac{\partial A_s}{\partial \phi}\right]\hat{z}$$

$$= \mu_0 n I \hat{z}$$

$$\Rightarrow \quad \frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} = 0$$

$$\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} = 0$$

$$\frac{1}{s}\left[\frac{\partial}{\partial s}(s A_\phi) - \frac{\partial A_s}{\partial \phi}\right] = \mu_0 n I$$

e) Then the differential equations become

$$\frac{\partial A_\phi}{\partial z} = 0 \quad \Rightarrow \quad A_\phi = A_\phi(s, \phi)$$

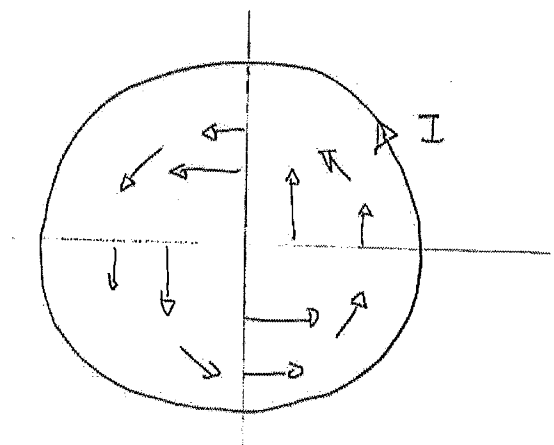
$$\frac{1}{s}\frac{\partial}{\partial s}(s A_\phi) = \mu_0 n I \quad \Rightarrow s A_\phi = \frac{\mu_0 n I s^2}{2} + f(\phi)$$

$$\Rightarrow \quad A_\phi = \frac{\mu_0 n I}{2} s + \frac{1}{s} f(\phi)$$

The simplest solution has

$$\vec{A} = \frac{\mu_0 n I}{2} s \, \hat{\phi}$$

We see that the vector potential circles in the same direction as the current.



Thus we have seen that:

1) A vector potential $\vec{A}$ always exists such that

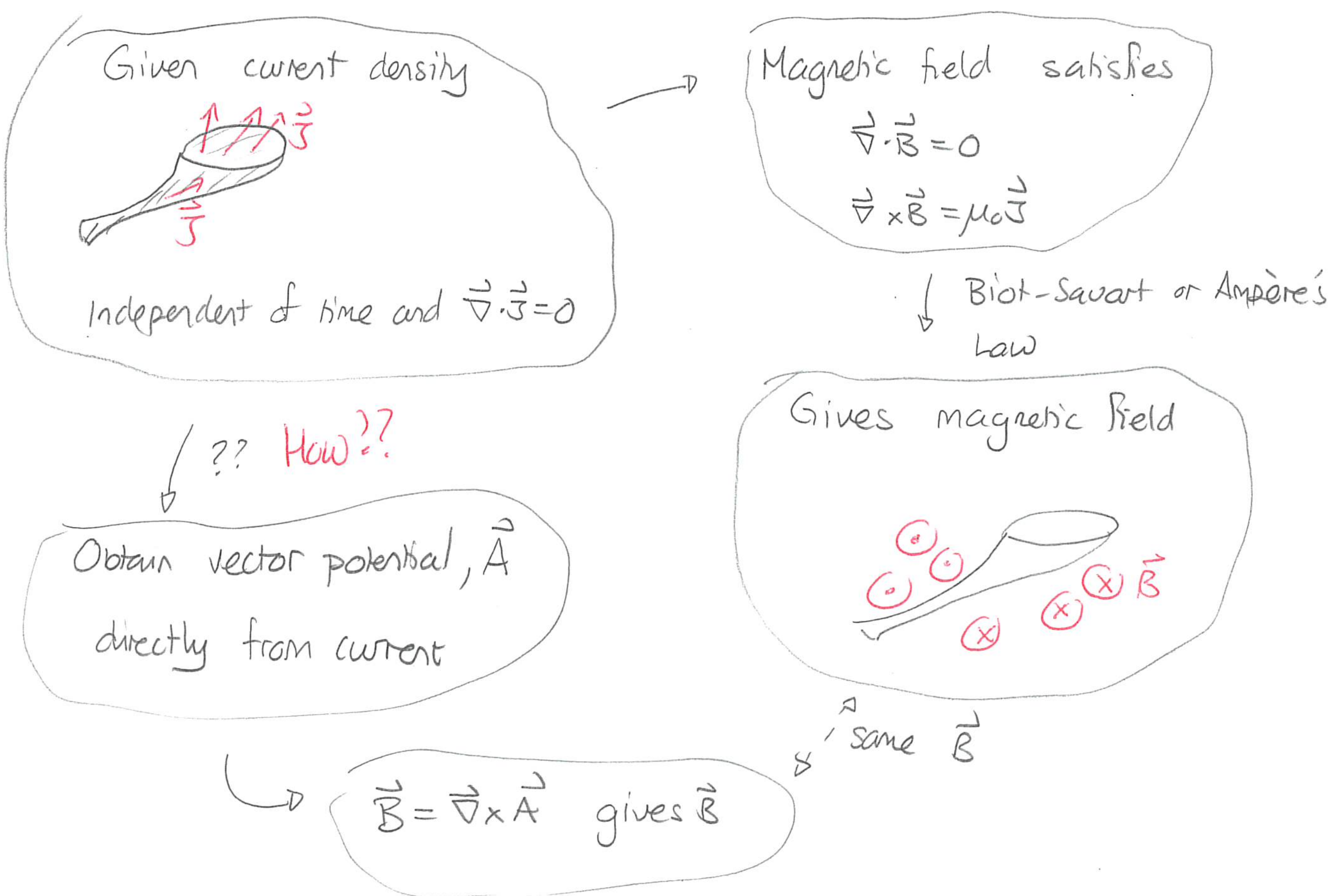
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

2) there will always be many possible vector potentials that can give the same field.

3) in at least some cases the vector potential has a similar direction to the current.

Vector potential from current density

We would like to be able to determine the vector potential from the current in a fashion similar to obtaining electrostatic potential from charge density via Poisson's equation.



The basic equation that relates $\vec{A}$ to $\vec{J}$ is obtained via

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

which translates to differential equations:

$$\frac{\partial}{\partial x}\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] - \left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right] = \mu_0 J_x$$

$$\vdots$$

In principle we have to solve these for a given $\vec{J}$ to determine $\vec{A}$. But can they be simplified via a choice of the possible potentials

<u>Choice of vector potential</u>

There are many vector potentials that can result in the same field and the range of choices is larger than that for electrostatic potentials. Note:

> If $\vec{A}$ generates a magnetic field $\vec{B}$ then, given <u>any scalar function</u>, λ,
>
> $$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$
>
> generates the same field.

<u>Proof:</u>

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\lambda)$$

$$= \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \vec{\nabla}\lambda}_{=0}$$

$$= \vec{\nabla} \times \vec{A} = \vec{B} \qquad \boxtimes$$

The other necessary fact is that:

$$\boxed{\begin{array}{l}\text{If } \vec{A} \text{ and } \vec{A}' \text{ generate the same field and } \alpha, \beta \text{ are} \\ \text{scalar constants that satisfy } \alpha+\beta=1, \text{ then} \\ \qquad \alpha \vec{A} + \beta \vec{A}' \\ \text{generates the same field as either } \vec{A} \text{ or } \vec{A}'.\end{array}}$$

Proof: $\vec{\nabla} \times (\alpha \vec{A} + \beta \vec{A}') = \alpha \underbrace{(\vec{\nabla} \times \vec{A})}_{\vec{B}} + \beta \underbrace{(\vec{\nabla} \times \vec{A}')}_{\vec{B}}$

$$= (\alpha+\beta) \vec{B} = \vec{B} \quad \text{if } \alpha+\beta=1 \quad \boxtimes$$

For example, with the solenoid:

$$\vec{A} = \mu_0 n I x \hat{y}$$

$$\vec{A}' = -\mu_0 n I y \hat{x}$$

and $\frac{1}{2}\vec{A} + \frac{1}{2}\vec{A}' = \frac{\mu_0 n I}{2}(x\hat{y} - y\hat{x}) = \frac{\mu_0 n I}{2} s \hat{\phi}$

produces the same magnetic field. An important fact about this example is $\vec{\nabla} \cdot (\frac{1}{2}\vec{A} + \frac{1}{2}\vec{A}') = 0$. Is it always possible to find a divergence-free vector potential?

Divergenceless vector potential

In general the vector potential must satisfy:

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

Suppose that $\vec{\nabla}\cdot\vec{A} \neq 0$. This is some function of position. Now consider

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

for some function λ. This generates the same magnetic field as $\vec{A}$. The divergence of $\vec{A}'$ is:

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda$$

If we can find λ such that

$$\nabla^2\lambda = -\underbrace{\vec{\nabla}\cdot\vec{A}}_{\text{function of position}} \iff \frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2} + \frac{\partial^2\lambda}{\partial z^2} = \underbrace{-\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y} - \frac{\partial A_z}{\partial z}}_{\text{function of } x,y,z}$$

then $\vec{\nabla}\cdot\vec{A}' = 0$. We can find such a λ, by solving Poisson's equation

$$\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}$$

Then we have:

> It is always possible to find a magnetic vector potential so that
>
> $$\vec{\nabla}\cdot\vec{A} = 0$$
>
> and, in this case,
>
> $$\nabla^2\vec{A} = -\mu_0\vec{J}$$

Now consider this vector equation. It gives

$$\nabla^2 A_x = -\mu_0 J_x$$

$$\nabla^2 A_y = -\mu_0 J_y$$

$$\nabla^2 A_z = \mu_0 J_z$$

which are three uncoupled Poisson equations. These can be solved by analogy with electrostatics:

$$\nabla^2 V = -\rho/\epsilon_0 \quad \Rightarrow \quad V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

Thus we get:

$$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

$$A_y = \frac{\mu_0}{4\pi}\int \frac{J_y(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

$$A_z = \frac{\mu_0}{4\pi}\int \frac{J_z(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

or, more compactly:

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'}$$