Fri: HW

Fri: Read 5.4.1

Ampère's Law

Ampère's law converts the statement $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ into an integral statement:

Consider any closed path.

Then

$$\oint_{\text{path}} \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

where the enclosed current is

$$I_{enc} = \int_{\text{surface}} \vec{J} \cdot d\vec{a}$$

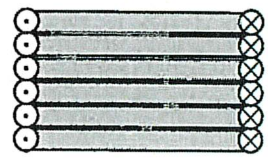
and the surface is any surface bounded by the closed path.

Here $\vec{J}$ is the source current density for the magnetic field

We can use this to compute magnetic fields in highly symmetric situations

1 Magnetic field produced by a solenoid

A solenoid is a stacked column of circular coils which carry current. Suppose that this is oriented with its axis along the z axis. A cross-sectional view is illustrated. Suppose that the number of loops per unit length is n and each carries current I.



a) Show that the magnetic field has form $\mathbf{B} = B_z(s)\hat{\mathbf{z}}$.

b) Show that the magnetic field is uniform inside the cylinder.

c) Show that the magnetic field is uniform outside the cylinder.

d) Determine the difference between the field inside and outside the cylinder.

e) Determine the field at all points.

Answer: a) In general

$$\vec{B} = B_s\,\hat{s} + B_\phi\,\hat{\phi} + B_z\,\hat{z}$$

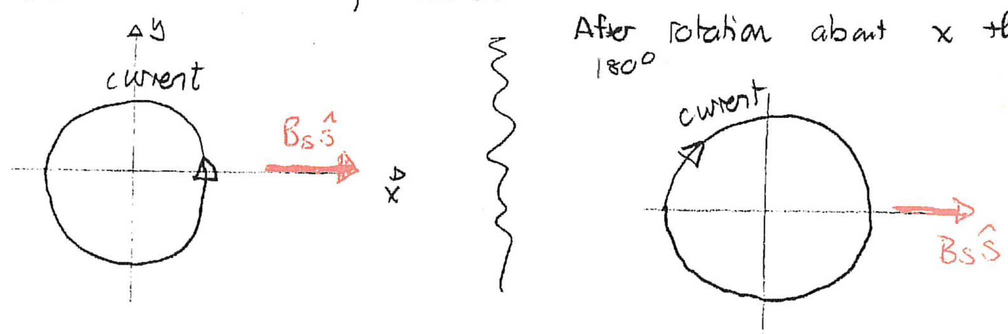
By symmetry the components cannot depend on z or ϕ. Thus

$$\vec{B} = B_s(s)\,\hat{s} + B_\phi(s)\,\hat{\phi} + B_z(s)\,\hat{z}$$

We will now eliminate all but $B_z(s)$. First consider $B_s(s)$.

Viewed down the loop axis:

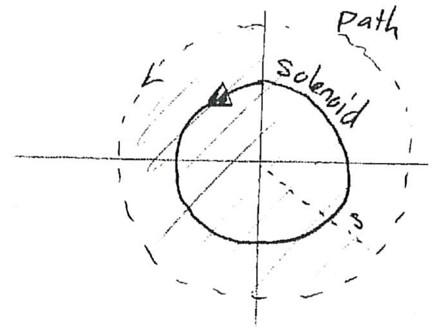
After rotation about x through 180°



This rotation reverses the current but not the s component of $\vec{B}$ which should be reversed if the current is reversed. So

$$B_s(s) = 0$$

Now consider $B_\phi(s)$. We can integrate around a circular path in the xy plane. Then Ampère's law gives:



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

But the current passing through the surface is zero. So

$$\oint \vec{B} \cdot d\vec{l} = 0$$

$$\Rightarrow \oint_{path} \left[B_\phi(s)\,\hat{\phi} + B_z(s)\,\hat{z} \right] s\, d\phi\, \hat{\phi} = 0$$

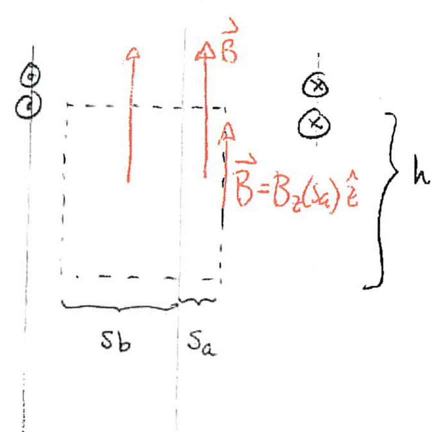
$$\Rightarrow \int_0^{2\pi} B_\phi(s)\, s\, d\phi = 0 \quad \Rightarrow \quad 2\pi s\, B_\phi(s) = 0$$

$$\Rightarrow B_\phi(s) = 0.$$

Thus

$$\vec{B} = B_z(s)\,\hat{z}$$

b) Consider a rectangular loop within the cylinder as oriented. Then



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = 0$$

$$\underbrace{\cancel{\int \vec{B} \cdot d\vec{l}}}_{top} + \underbrace{\int \vec{B} \cdot d\vec{l}}_{left} + \underbrace{\cancel{\int \vec{B} \cdot d\vec{l}}}_{bottom} + \underbrace{\int \vec{B} \cdot d\vec{l}}_{right} = 0$$

0 perpendicular. 0

$$\int_{left} \vec{B} \cdot d\vec{l} + \int_{right} \vec{B} \cdot d\vec{l} = 0$$

On the right side

$$\vec{B} = B_z(s_a)\hat{z} \qquad \vec{dl} = dz\,\hat{z}$$

$$\Rightarrow \int_{\text{right}} \vec{B}\cdot\vec{dl} = B_z(s_a)\,h$$

On the left side $\vec{B} = B_z(s_b)\hat{z}$ $\quad \vec{dl} = -dz\,\hat{z}$

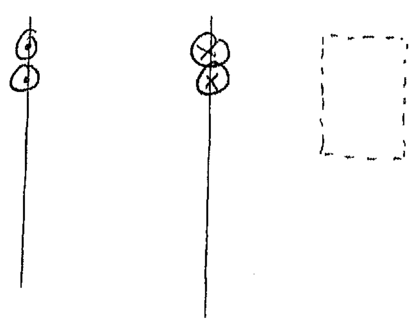
$$\Rightarrow \int \vec{B}\cdot\vec{dl} = -B_z(s_b)\,h$$

Thus $\qquad B_z(s_a)\,h - B_z(s_b)\,h = 0 \quad \Rightarrow \quad B_z(s_a) = B_z(s_b).$

This applies for all s_a, s_b. Thus $\vec{B}$ is uniform inside. So inside

$$\vec{B} = B_{in}\,\hat{z}$$

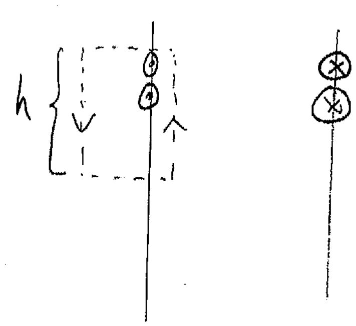
c) We can construct a similar loop outside the cylinder. Then the same argument applies



$$\vec{B} = B_{out}\,\hat{z}$$

uniform outside

d) Use a rectangular loop that straddles the solenoid. Then



$$\int \vec{B}\cdot\vec{dl} = \mu_0 I_{enc}$$

$$= \mu_0 I \times \text{number of loops}$$

$$= \mu_0 I n h$$

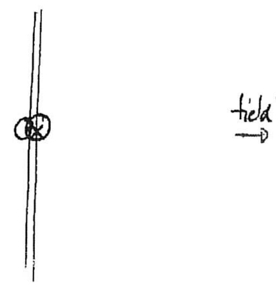
On the left hand side

$$\oint \vec{B} \cdot d\vec{l} = B_{in} h - B_{out} h$$

$$\Rightarrow (B_{in} - B_{out}) \cancel{h} = \mu_0 I n \cancel{h}$$

$$\Rightarrow B_{in} - B_{out} = \mu_0 n I$$

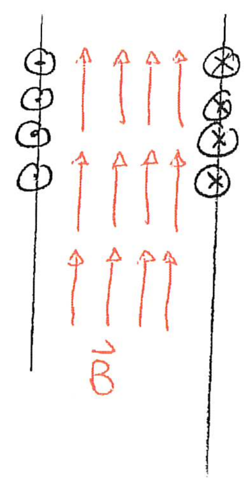
e) We can consider the field infinitely far from the solenoid. Here it will appear that the current is effectively zero since currents in/out nearly overlap. Thus the field approaches zero at infinite distances. So $B(s \to \infty) = 0$. But the field is uniform outside the solenoid. So $B_{out} = 0$



Then $B_{in} - B_{out} = \mu_0 n I \Rightarrow B_{in} = \mu_0 n I$

Thus we get

$$\vec{B} = \begin{cases} \mu_0 n I \hat{z} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

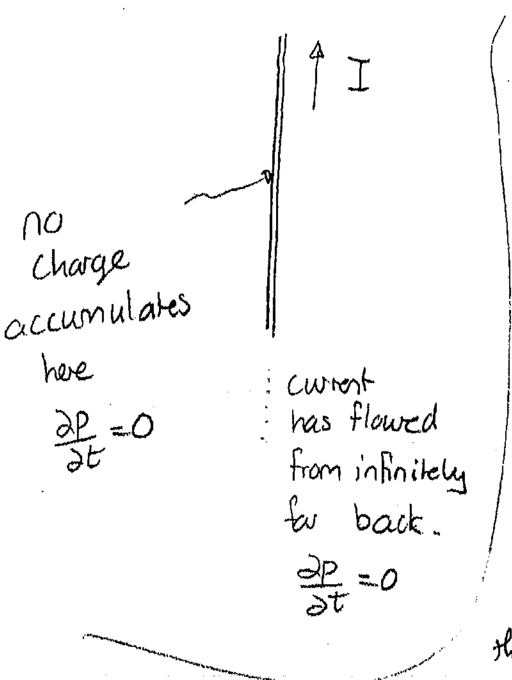


Limitations of Ampère's Law

The derivation of Ampère's law requires that $\vec{\nabla}\cdot\vec{J}=0$. There is separately a general law connecting current density to the density of any charges that are involved in the current and this requires $\frac{\partial\rho}{\partial t}=0$. So for Ampère's law to be valid we need

$$\vec{\nabla}\cdot\vec{J}=0 \quad \text{and} \quad \frac{\partial\rho}{\partial t}=0$$

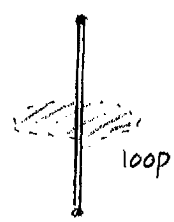
In the case of the infinite straight current both of these are true. Now

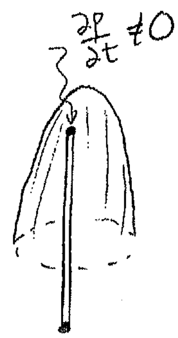


consider a finite current segment. We can then construct an Ampèrian loop around the current. But the same loop can be bounded by various surfaces. If Ampère's law is true

$$\oint\vec{B}\cdot d\vec{l}=\mu_0 I_{enc}$$

must apply for both surfaces. However, the l.h.s must give the same result regardless of the surface.





For the disk surface $\quad I_{enc}=I \quad \Rightarrow \quad \oint\vec{B}\cdot d\vec{l}=\mu_0 I$

For the dome surface $\quad I_{enc}=0 \quad \Rightarrow \quad \oint\vec{B}\cdot d\vec{l}=0$

This is a contradiction. The source is that if current is to flow the charge density at the ends must change with time. Thus $\frac{\partial\rho}{\partial t}\neq 0$ at the ends and Ampère's law is not applicable

Magnetic Vector Potential

For magnetostatic fields $\vec{\nabla} \cdot \vec{B} = 0$. We know from vector calculus that for any vector field $\vec{A}$,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

So the question is

1) does there exist a vector field $\vec{A}$ so that

$$\vec{B} = \vec{\nabla} \times \vec{A} \; ?$$

2) if such a field exists can we determine $\vec{A}$ directly from the current density, $\vec{J}$, for the source currents for the fields?

If such a field exists then we call this the magnetic vector potential. In general

> If a magnetic vector potential $\vec{A}$ exists then the associated magnetic field is
>
> $$\vec{B} = \vec{\nabla} \times \vec{A}$$

2 Vector potential for an infinitely long straight wire

An infinitely long wire along the z axis carries steady current I. The magnetic field produced by this, in terms of cylindrical coordinates, is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}.$$

a) The magnetic vector potential for this can be expressed as

$$\mathbf{A} = A_s\hat{\mathbf{s}} + A_\phi\hat{\phi} + A_z\hat{\mathbf{z}}.$$

Determine a set of differential equations that the components of the vector potential satisfy.

b) Find a solution that satisfies $A_\phi = A_z = 0$. Sketch the resulting vector potential.

c) Find a solution that satisfies $A_\phi = A_s = 0$. Sketch the resulting vector potential.

Answer: a) $\vec{\nabla}\times\vec{A} = \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$

In cylindrical co-ordinates,

$$\vec{\nabla}\times\vec{A} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s A_\phi) - \frac{\partial A_s}{\partial\phi}\right]\hat{z}$$

Comparing gives:

$$\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} = 0$$

$$\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} = \frac{\mu_0 I}{2\pi s}$$

$$\frac{\partial}{\partial s}(s A_\phi) - \frac{\partial A_s}{\partial\phi} = 0$$

b) In this case

$$0 = 0$$

$$\frac{\partial A_s}{\partial z} = \frac{\mu_0 I}{2\pi s}$$

$$-\frac{\partial A_s}{\partial \phi} = 0 \qquad \Rightarrow \qquad A_s = A_s(s, z).$$

So

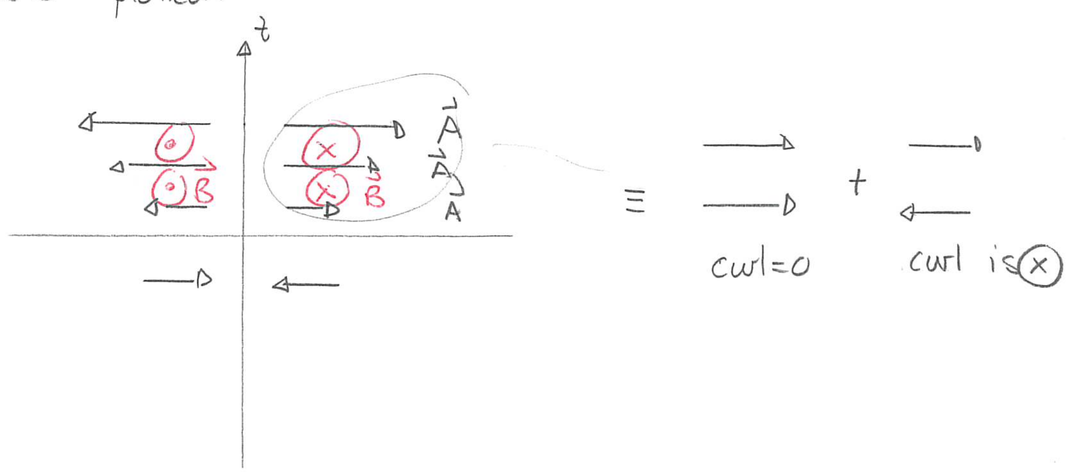
$$\frac{\partial A_s}{\partial z} = \frac{\mu_0 I}{2\pi s} \Rightarrow A_s = \frac{\mu_0 I}{2\pi s} z + f(s)$$

where $f(s)$ is an arbitrary function. Thus

$$\vec{A} = \left[\frac{\mu_0 I}{2\pi} \frac{z}{s} + f(s)\right] \hat{s}$$

Consider $f(s) \equiv 0$. Then $\vec{A} = \frac{\mu_0 I}{2\pi} \frac{z}{s} \hat{s}$

This is plotted:



c) In this case

$$\frac{\partial A_z}{\partial \phi} = 0 \quad \Rightarrow \quad A_z = A_z(s, z)$$

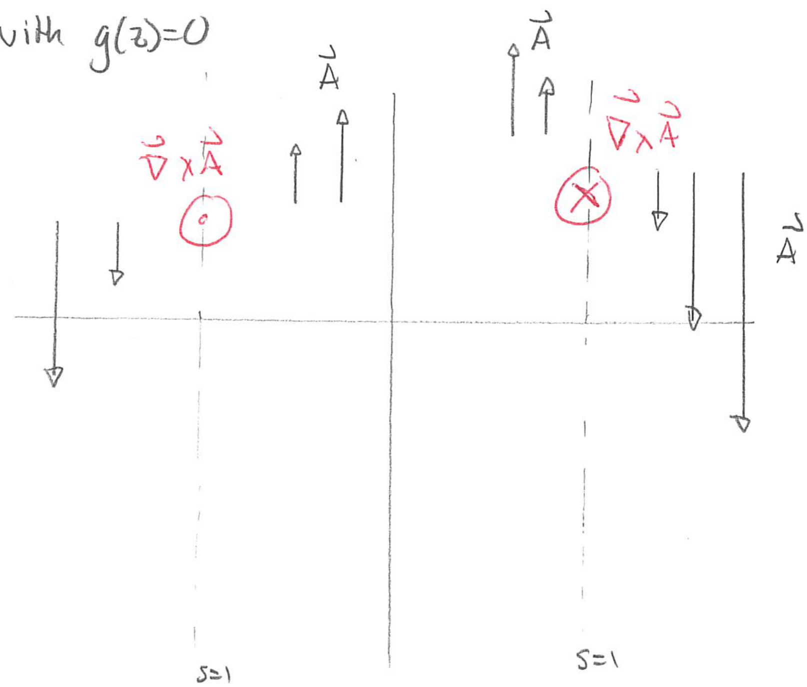
$$-\frac{\partial A_z}{\partial s} = \frac{\mu_0 I}{2\pi s} \quad \Rightarrow \quad A_z = -\frac{\mu_0 I}{2\pi} \ln(s) + g(z)$$

where $g(z)$ is arbitrary.

So

$$\vec{A} = \left[-\frac{\mu_0 I}{2\pi} \ln(s) + g(z) \right] \hat{z}$$

With $g(z)=0$



This example illustrates the facts:

1) there exist possible vector potentials $\vec{A}$ so that $\vec{B} = \vec{\nabla} \times \vec{A}$

2) for any given field $\vec{B}$, there are many possible vector potentials whose curls give the same field. These differ by entire functions of one or more co-ordinates