Tues HW 18

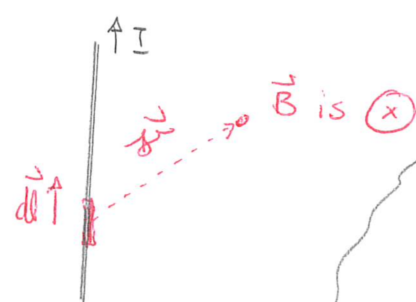
Weds: Read

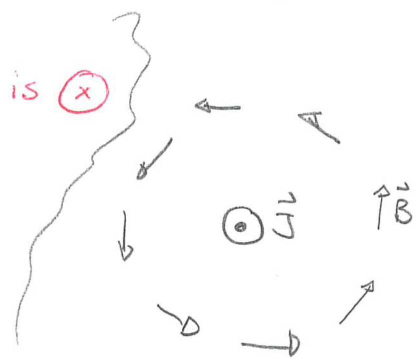
Magnetic Field Differential Equations

The basic law for electrostatic fields, Coulomb's Law could be used to derive two sets of differential equations that all electrostatic fields must satisfy:

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \text{and} \qquad \vec{\nabla}\times\vec{E} = 0 .$$

Presumably a similar process applies to the Biot-Savart Law. To obtain insight into the resulting differential equations for magnetic fields, consider the field produced by a straight current. The Biot-Savart law indicates that this field will circle the current. Clearly then $\vec{\nabla}\times\vec{B} \neq 0$, and may be related to the current density direction.





The comparable rule for $\vec{\nabla}\cdot\vec{B}$ is not immediately clear although the example suggests that $\vec{\nabla}\cdot\vec{B} = 0$.

Applying vector calculus gives:

> For any magnetostatic field (produced by a stationary current)
>
> $$\vec{\nabla}\cdot\vec{B} = 0$$
>
> $$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$
>
> where $\vec{J}$ is the source current density for the field.

Proof: First consider $\vec{\nabla} \cdot \vec{B}$

The Biot-Savart law gives

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times \hat{\gamma}}{\gamma^2} d\tau'$$

and

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

Thus

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \cdot \frac{\vec{J}(\vec{r}\,') \times \hat{\gamma}}{\gamma^3} d\tau'$$

Then $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$. So

$$\vec{\nabla} \cdot \frac{\vec{J}(\vec{r}\,') \times \hat{\gamma}}{\gamma^2} = \frac{\hat{\gamma}}{\gamma^2} \cdot \vec{\nabla} \times \vec{J}(\vec{r}\,') - \vec{J}(\vec{r}\,') \cdot \left(\vec{\nabla} \times \frac{\hat{\gamma}}{\gamma^2}\right)$$

Now $\vec{\nabla}$ differentiates w.r.t x, y, z and $\vec{J}(\vec{r}\,')$ is a function of x', y', z'. Thus $\vec{\nabla} \times \vec{J}(\vec{r}\,') = 0$.

Then

$$\vec{\nabla} \times \frac{\hat{\gamma}}{\gamma^2} = \vec{\nabla} \times \left[\frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \right.$$

$$= \hat{x} \left\{ \frac{\partial}{\partial y} \frac{(z-z')}{[\ldots]^{3/2}} - \frac{\partial}{\partial z} \frac{(y-y')}{[\ldots]^{3/2}} \right\} + \hat{y} \{ \ldots$$

$$= \hat{x}\, 0 + \hat{y}\, 0 + \hat{z}\, 0.$$

Thus

$$\vec{\nabla} \cdot \frac{J(\vec{r}\,') \times \hat{\gamma}}{\gamma^2} = 0 \qquad \Rightarrow \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Second consider $\vec{\nabla}\times\vec{B}$.

We proceed as before giving

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\right)d\tau'$$

Now the integrand has form

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

Here $\vec{A} \equiv \vec{J}(\vec{r}\,')$ and since differentiation is w.r.t unprimed

$$\vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}\,')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\right) = -\left[\vec{J}(\vec{r}\,')\cdot\vec{\nabla}\right]\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} + \vec{J}(\vec{r}\,')\underbrace{\left(\vec{\nabla}\cdot\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right)}_{4\pi\,\delta(\vec{\mathscr{r}})}$$

Now

$$\vec{A}\cdot\vec{\nabla}\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right) = -\vec{A}\cdot\vec{\nabla}'\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right)$$

where $\vec{\nabla}' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}$. So

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right)d\tau' + \frac{\mu_0}{\cancel{4\pi}}\cancel{4\pi}\int \vec{J}(\vec{r}\,')\underbrace{\delta(\vec{\mathscr{r}})}_{\delta(\vec{r}-\vec{r}\,')}d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right)d\tau' + \mu_0\underbrace{\int \vec{J}(\vec{r}\,')\,\delta(\vec{r}-\vec{r}\,')\,d\tau'}_{\vec{J}(\vec{r})}$$

Then $$\vec{A}\cdot\vec{\nabla}'\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \vec{A}\cdot\vec{\nabla}'\frac{x-x'}{[\ldots]^{3/2}}\,\hat{x} + \cdots$$

$$= \left\{\vec{A}\cdot\vec{\nabla}'\frac{x-x'}{[\ldots]^{3/2}}\right\}\hat{x} + \cdots$$

But $\vec{\nabla}\cdot(f\vec{A}) = \vec{\nabla}f\cdot\vec{A} + f\,\vec{\nabla}\cdot\vec{A}$ gives:

$$\vec{A}\cdot\vec{\nabla}f = \vec{\nabla}\cdot(f\vec{A}) - f(\vec{\nabla}\cdot\vec{A})$$

So

$$\vec{A}\cdot\vec{\nabla}'\frac{x-x'}{\mathscr{r}^3} = \vec{\nabla}'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{A}\right] - \frac{x-x'}{\mathscr{r}^3}\vec{\nabla}'\cdot\vec{A}$$

Thus

$$\vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{x-x'}{\mathscr{r}^3}\right) = \vec{\nabla}'\cdot\left[\frac{x-x'}{\mathscr{r}^3}\vec{J}(\vec{r}')\right] - \frac{x-x'}{\mathscr{r}^3}\vec{\nabla}'\cdot\vec{J}(\vec{r}')$$

Now one can show that in general a current density and the associated charge density satisfy:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

and in steady state situations $\frac{\partial\rho}{\partial t} = 0$. So we get

$$\vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{x-x'}{\mathscr{r}^3}\right) = \vec{\nabla}'\cdot\left[\frac{x-x'}{\mathscr{r}^3}\vec{J}(\vec{r}')\right].$$

Thus

$$\int(\vec{J}(\vec{r}')\cdot\vec{\nabla}')\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}d\tau' = \int\left(\vec{\nabla}'\cdot\left[\frac{x-x'}{\mathscr{r}^3}\vec{J}(\vec{r}')\right]\hat{x} + \dots\right)d\tau'$$

Thus can be converted into three surface integrals

$$\int\vec{J}(\vec{r}')\cdot\vec{\nabla}'\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}d\tau' = \iint\left[\frac{x-x'}{\mathscr{r}^3}\vec{J}(\vec{r}')\cdot d\vec{a}'\right]\hat{x} \quad + \dots$$

For localized currents $\vec{J}(\vec{r}')\cdot d\vec{a}' \to 0$ for infinitely large surfaces. Thus

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})$$

We thus find that we need to solve the following partial differential equations:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 J_y$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 J_z$$

We will present two strategies for solving these:

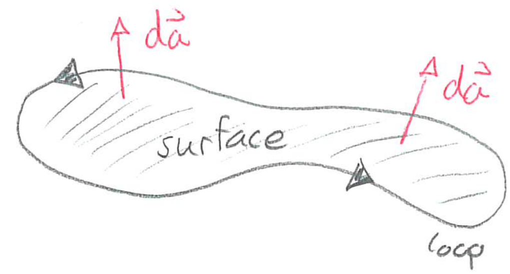
1) Rewrite $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ so that it will enable a direct calculation of $\vec{B}$.

2) Rework $\vec{\nabla} \cdot \vec{B} = 0$ to arrive at a potential that can yield $\vec{B}$ via differentiation.

Ampère's Law

Consider the equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$. We can use Stoke's Law to rewrite this in an integral form. This requires a closed loop. Then

$$\int_{\text{surface}} \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \oint_{\text{loop}} \vec{B} \cdot d\vec{l}$$



$$\parallel$$

$$\mu_0 \int \vec{J} \cdot d\vec{a} = \oint_{\text{loop}} \vec{B} \cdot d\vec{l}$$

Note consistency of $d\vec{a}$ with loop sense. Using r.h. follow loop with fingers $\rightarrow$ thumb gives $d\vec{a}$.

Now $\int \vec{J} \cdot d\vec{a}$ is the current passing through the surface enclosed by the loop. Thus we arrive at the integral form of Ampère's Law:

> Consider a magnetostatic field $\vec{B}$. Then for any closed path,
>
> $$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$
>
> where
>
> $$I_{enc} = \int \vec{J} \cdot d\vec{a}$$
>
> is the source current that passes through the surface enclosed by the loop.

We will see that, similar to Gauss' Law this will facilitate calculation of magnetic fields in symmetric situations

1 Field produced by an infinitely long wire

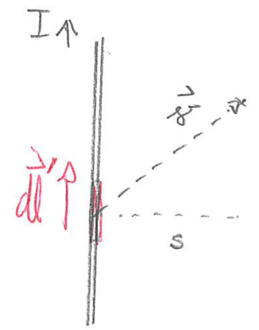
An infinitely long wire carries current I.

a) Use the Biot-Savart law to determine the direction of the field produced by the current and express the result in cylindrical coordinates. Use symmetry to further constrain the field.

b) Consider a circular loop of radius s, centered on the wire and in a plane parallel to the wire. Evaluate $\oint \mathbf{B} \cdot \mathbf{dl}$ for this loop.

c) Use Ampère's law to determine the magnetic field produced by the current.

Answer: a)



The Biot-Savart law gives

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

Then $d\vec{l}' \times \hat{\mathscr{r}}$ is into / out of page

Thus $d\vec{B}$ is along $\hat{\phi}$.

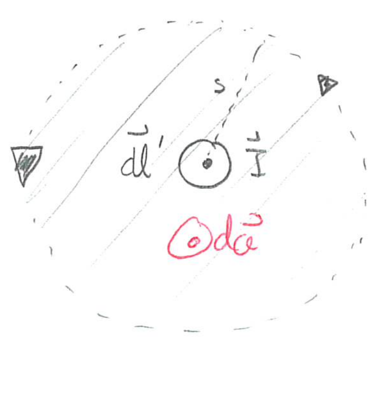
A separate derivation uses

$$\left.\begin{aligned} d\vec{l}' &= dl'\hat{z} \\ \vec{\mathscr{r}} &= \mathscr{r}_s \hat{s} + \mathscr{r}_z \hat{z} \end{aligned}\right\} \Rightarrow d\vec{l}' \times \vec{\mathscr{r}} = \ldots \underbrace{\hat{z} \times \hat{s}}_{\hat{\phi}} = \ldots \hat{\phi}$$

By symmetry $\vec{B}$ is also independent of ϕ and z. Thus

$$\vec{B} = B_\phi(s)\,\hat{\phi}$$

b) Consider a circular loop centered on the current. Viewed from above:



Then along the loop

$s' = \text{constant} = s$

$0 \leq \phi' \leq 2\pi$

$z' = \text{constant}$

$$d\vec{l}' = s'd\phi'\,\hat{\phi} = s\,d\phi'\,\hat{\phi}$$

$$\Rightarrow \quad \vec{B}\cdot d\vec{l}' = B_\phi(s)\, s\, d\phi'$$

$$\Rightarrow \quad \oint \vec{B}\cdot d\vec{l}' = \int_0^{2\pi} \underbrace{B_\phi(s)\, s}_{\text{constant.}}\, d\phi' = 2\pi s\, B_\phi(s)$$

c) By Ampère's Law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

where I_{enc} is the current passing through the shaded loop in the $d\vec{a}$ direction. Here $I_{enc} = I$. Thus

$$2\pi s\, B_\phi(s) = \mu_0 I$$

$$\Rightarrow \quad B_\phi(s) = \frac{\mu_0 I}{2\pi s}$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi} \qquad \boxtimes$$

The Biot-Savart law would yield the same result by direct integration

2 Field produced by an infinitely long cylinder

An infinitely long wire cylinder oriented along the z axis with radius R carries current with density

$$\mathbf{J} = \alpha s'^2\, \hat{\mathbf{z}}$$

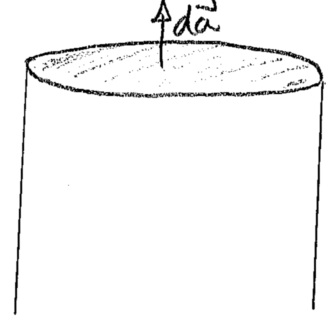
where α has units of $\mathrm{A/m^4}$.

a) Determine an expression for α in terms of the total current flowing down the cylinder.

b) Use Ampère's law to determine the magnetic field produced by the current *at all points.*

Answer: a) Consider integration over the "end" of the cylinder

$$I = \int_{\text{cylinder end}} \vec{J}\cdot d\vec{a}$$



Here $0 \le s' \le R$

$0 \le \phi' \le 2\pi$

$z' = \text{const}$

and $d\vec{a} = s'ds'd\phi'\,\hat{z}$

So $\vec{J}\cdot d\vec{a} = \alpha s'^2\, s'ds'd\phi' = \alpha s'^3 ds' d\phi'$

and

$$I = \int_0^R ds' \int_0^{2\pi} d\phi'\, \alpha s'^3 = \alpha \underbrace{\int_0^R s'^3 ds'}_{R^4/4} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi}$$

$$\Rightarrow I = \frac{\alpha R^4}{2}\pi \quad \Rightarrow \alpha = \frac{2I}{\pi R^4}$$

So $$\vec{J} = \frac{2I}{\pi R^4} s'^2\, \hat{z}$$

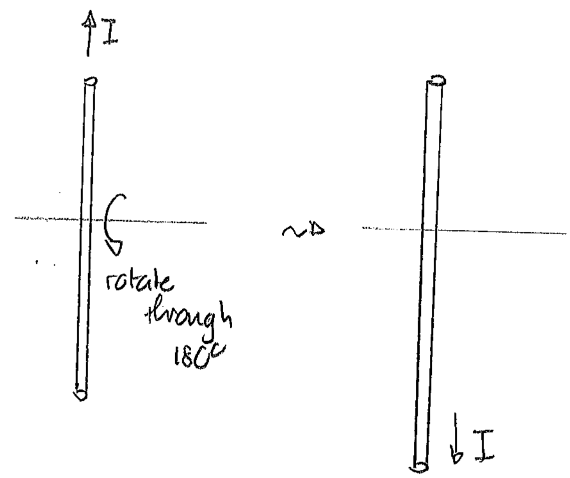
b) In general the field has the form

$$\vec{B} = B_s \hat{s} + B_\phi \hat{\phi} + B_z \hat{z}$$

and we need to constrain the components. We use a consequence of the Biot-Savart Law:

If a source current is reversed then the field reverses

So consider rotating the cylinder about an axis perpendicular to the cylinder axis. Then



$$\vec{B} \rightarrow -B_s \hat{s} - B_\phi \hat{\phi} - B_z \hat{z}$$

However the physical rotation would only flip B_ϕ and B_z

$$\vec{B} \rightarrow B_s \hat{s} - B_\phi \hat{\phi} - B_z \hat{z}$$

Thus $B_s = 0$

Now the Biot-Savart Law also implies that the field is perpendicular to $\vec{J}$, or $\vec{I}$. So $B_z = 0$. Thus

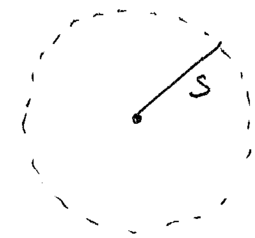
$$\vec{B} = B_\phi \hat{\phi}$$

By symmetry B_ϕ can only depend on s. So

$$\vec{B} = B_\phi(s) \hat{\phi}$$

To use Ampère's law we need a closed loop. Use a circle with radius s centered on the z axis. This will yield the field at distanse s from the axis. Regardless of the location of the circle inside or outside the cylinder the integral

$$\oint \vec{B} \cdot d\vec{l}'$$



will give the same result. Here

$$\left.\begin{array}{c} s' = s \\ 0 \leq \phi' \leq 2\pi \\ z' = \text{const} \end{array}\right\} \quad d\vec{l}' = s' d\phi' \hat{\phi} = s\, d\phi' \hat{\phi}$$

So

$$\oint \vec{B} \cdot d\vec{l}' = \int_0^{2\pi} B_\phi(s)\, s\, d\phi'$$

$$= 2\pi s\, B_\phi(s)$$

and Ampère's law gives:

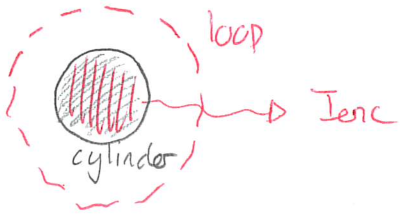
$$2\pi s\, B_\phi(s) = \mu_0 I_{enc}$$

$$\Rightarrow \boxed{B_\phi(s) = \frac{\mu_0}{2\pi} \frac{I_{enc}}{s}}$$

Here I_{enc} is the current passing through the loop of radius s.

There are two cases:

$s \geq R$

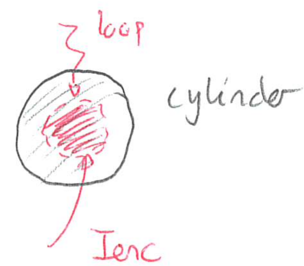


Here $I_{enc} = I$

$$\Rightarrow B_\phi(s) = \frac{\mu_0}{2\pi} \frac{I}{s}$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{2\pi} \frac{I}{s} \hat{\phi}$$

$s \leq R$



Here $I_{enc} = \int_{\text{inside } s} \vec{J} \cdot d\vec{a}$

$$\left.\begin{array}{c} 0 \leq s' \leq s \\ 0 \leq \phi' \leq 2\pi \\ z' = \text{const} \end{array}\right\} d\vec{a} = s' ds' d\phi' \hat{z}$$

$$\vec{J} \cdot d\vec{a} = \alpha s'^2 s' ds' d\phi'$$

$$= \alpha s'^3 ds' d\phi'$$

$$\int \vec{J} \cdot d\vec{a} = \alpha \int_0^s s'^3 ds' \int_0^{2\pi} d\phi'$$

$$= \frac{\alpha s^4}{4} 2\pi = \frac{\alpha s^4 \pi}{2}$$

$$= \frac{s^4}{R^4} I$$

Thus $B_\phi(s) = \frac{\mu_0}{2\pi} \frac{s^4 I}{R^4 s} = \frac{\mu_0}{2\pi} \frac{s^3}{R^4} I$

$$\Rightarrow \vec{B} = \frac{\mu_0}{2\pi} \frac{I}{R^4} s^3 \hat{\phi}$$

So

$$\vec{B} = \begin{cases} \frac{\mu_0}{2\pi} \frac{I}{R^4} s^3 \hat{\phi} & s \leq R \\ \frac{\mu_0}{2\pi} \frac{I}{s} \hat{\phi} & s \geq R \end{cases}$$