Phys 311 2020

Lecture 28

Fri: HW by 5pm

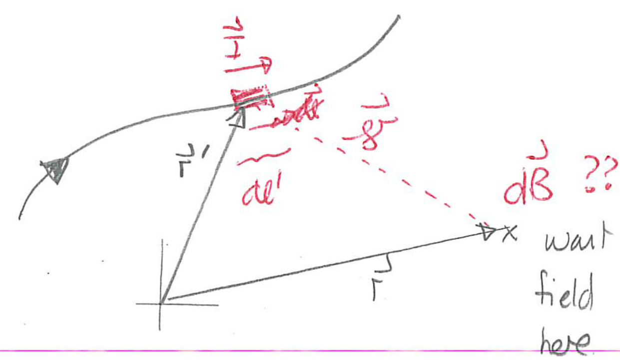
Mon. Read 5.3.1 → 5.3.3.

Biot-Savart Law

We will first consider how a stationary current produces a magnetic field. Consider a segment of current carrying wire. We will determine the contribution to the field from each infinitesimal portion of the wire. Then the total field will be

$$\vec{B} = \int d\vec{B}$$

contributions for all portions of wire



The fundamental rule for doing this refers to an infinitesimal segment. This contributes

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} dl'$$

where $\mu_0 = 4\pi \times 10^{-7}\ N/A^2$ (permittivity of free space)

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ (for the illustrated set up)

The units of the field are Teslas. $T = \frac{N}{A \cdot m}$

The entire segment of wire produces a field

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, dl'}$$

depend on $\vec{r}\,'$ and thus on integration variables (NOT CONSTANT)

Alternatively

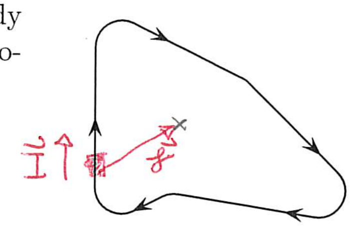
$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int I \, \frac{d\vec{l}\,' \times \hat{\mathscr{r}}}{\mathscr{r}^2}}$$

depend on primed co-ords

This is the Biot-Savart Law and is the fundamental rule that provides magnetic fields produced by stationary currents.

1 Magnetic field direction

An arbitrarily shaped loop lies in a plane and carries a steady current. Determine the direction of the magnetic field produced by the current at any point within the loop.



Answer: Consider the illustrated point and a portion of the current. For this

$\vec{I} \times \hat{r}$ is <u>into the page</u>

The same is true for any other segment. Every segment of current produces a field <u>into the page</u>. Adding gives

$\vec{B} \equiv$ into page

We can determine this via r.h. rules

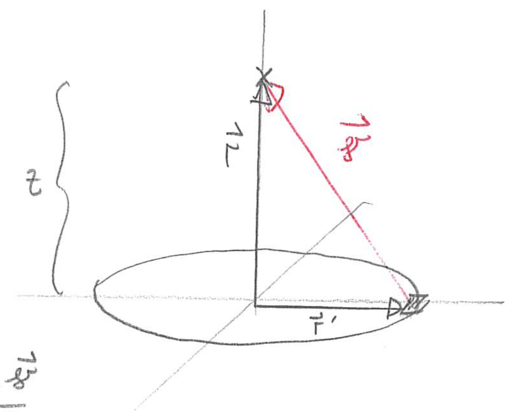
1) grab wire with r.h. and thumb along current fingers curl around with in sense of field

2) circle finger of r.h. in current sense. Thumb points along $\vec{B}$

2 Biot-Savart law for one dimensional currents

A circular loop with radius R carries a steady current I. Determine the magnetic field produced by the current at any point along the axis of the loop.

Answer: Orient the loop as illustrated.



We decompose + integrate:

Consider the contribution from the shaded segment:

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} I \frac{d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

To construct this:

$$\vec{r} = z\hat{z}$$

$$\vec{r}' = s'\hat{s} = R\hat{s}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = z\hat{z} - R\hat{s}$$

$$\mathscr{r} = \sqrt{\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}} = \left[(z\hat{z} - R\hat{s}) \cdot (z\hat{z} - R\hat{s})\right]^{1/2} = \sqrt{z^2 + R^2}$$

Then

$$d\vec{l}' = R\,d\phi'\,\hat{\phi} \qquad 0 \leq \phi' \leq 2\pi$$

So

$$d\vec{l}' \times \vec{\mathscr{r}} = R\,d\phi'\,\hat{\phi} \times \left[z\hat{z} - R\hat{s}\right]$$

$$= Rz\,d\phi'\,\underbrace{\hat{\phi} \times \hat{z}}_{\hat{s}} - R^2 d\phi'\,\underbrace{\hat{\phi} \times \hat{s}}_{-\hat{z}}$$

$$= Rz\,d\phi'\,\hat{s} + R^2 d\phi'\,\hat{z}$$

$$= R\left[z\,d\phi'\,\hat{s} + R\,d\phi'\,\hat{z}\right]$$

Thus

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{IR}{(R^2+z^2)^{3/2}}\left[z\hat{s} + R\hat{z}\right]d\phi'$$

Before integrating recognize that $\hat{s}$ varies around the loop

$$\hat{s} = \cos\phi'\,\hat{x} + \sin\phi'\,\hat{y}$$

Thus

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{IR}{(R^2+z^2)^{3/2}}\left[z\cos\phi'\,\hat{x} + z\sin\phi'\,\hat{y} + R\hat{z}\right]d\phi'$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0}{4\pi}\frac{IR}{(R^2+z^2)^{3/2}}\int_0^{2\pi}\left[z\cos\phi'\,\hat{x} + z\sin\phi'\,\hat{y} + R\hat{z}\right]d\phi'$$

$$= \frac{\mu_0}{4\pi}\frac{IR}{(R^2+z^2)^{3/2}}\left\{\overbrace{\cancel{\int_0^{2\pi} z\cos\phi'\,d\phi'}}^{0}\,\hat{x} + \overbrace{\cancel{\int_0^{2\pi} z\sin\phi'\,d\phi'}}^{0}\,\hat{y} + \int_0^{2\pi} R\hat{z}\,d\phi'\right]$$

$$= \frac{\mu_0}{4\pi}\frac{IR}{(R^2+z^2)^{3/2}}\,R\,2\pi\,\hat{z}$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0 I R^2}{2\,(R^2+z^2)^{3/2}}\,\hat{z}$$

The Biot - Savart law generalizes to two and three dimensional current distributions. For a two dimensional current:

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int_{\text{surface}} \frac{\vec{K} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, da'}$$

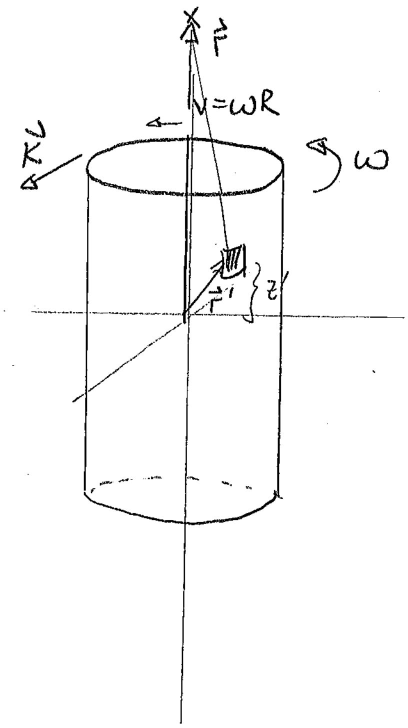
For a three dimensional current:

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int_{\substack{\text{volume} \\ \text{region}}} \frac{\vec{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'}$$

3 Biot-Savart law for two dimensional currents

A hollow cylindrical shell with radius R and height h carries a uniform surface charge density σ. It rotates with constant angular velocity ω. Determine the magnetic field produced at any point along the axis of the cylinder. This will require the integral

$$\int \frac{1}{(a^2+u^2)^{3/2}}\, \mathrm{d}u = \frac{u}{a^2\,(a^2+u^2)^{1/2}}.$$



Answer: The rotating cylinder produces a surface current

$$\vec{K} = \sigma \vec{v}$$

Then $\vec{v} = \omega R\,\hat{\phi}$

$$\Rightarrow \quad \vec{K} = \sigma\omega R\,\hat{\phi}$$

We consider a field point

$$\vec{r} = z\hat{z}$$

Then consider the field contribution from the shaded point. This is at

$$\vec{r}' = s\hat{s} + z'\hat{z}$$
$$= R\hat{s} + z'\hat{z}$$

$$\Rightarrow \quad \vec{\mathscr{r}} = \vec{r} - \vec{r}' = (z-z')\hat{z} - R\hat{s}$$

again $\mathscr{r} = \left[(z-z')^2 + R^2\right]^{1/2}$. Then

$$d\vec{B} = \frac{\mu_0}{4\pi}\,\frac{\vec{K}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,da' = \frac{\mu_0}{4\pi}\,\frac{\vec{K}\times\vec{\mathscr{r}}}{\mathscr{r}^3}\,da'$$

requires da'

For this

$$\left.\begin{aligned} s' &= R \\ 0 \le \phi' &\le 2\pi \\ -h/2 \le z' &\le h/2 \end{aligned}\right\} \quad \begin{aligned} da' &= s'd\phi'dz' \\ &= R\,d\phi'dz' \end{aligned}$$

Then

$$d\vec{B} = \frac{\mu_0}{4\pi}\sigma\omega R \frac{\hat{\phi}\times\left[(z-z')\hat{z} - R\hat{s}\right]}{\left[(z-z')^2 + R^2\right]^{3/2}} R\,d\phi'dz'$$

$$= \frac{\mu_0}{4\pi}\sigma\omega R^2 \frac{(z-z')\hat{s} + R\hat{z}}{\left[(z-z')^2 + R^2\right]^{3/2}} d\phi'dz'$$

Again $\hat{s} = \cos\phi'\hat{x} + \sin\phi'\hat{y}$ gives:

$$d\vec{B} = \frac{\mu_0}{4\pi}\sigma\omega R^2 \frac{(z-z')\cos\phi'\hat{x} + (z-z')\sin\phi'\hat{y} + R\hat{z}}{\left[(z-z')^2 + R^2\right]^{3/2}} d\phi'dz'$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0}{4\pi}\sigma\omega R^2 \left\{ \int_{-h/2}^{h/2} dz' \int_0^{2\pi} d\phi' \overbrace{\cancel{\frac{(z-z')\cos\phi'}{[\ldots]^{3/2}}}}^{0}\hat{x} + \int_{-h/2}^{h/2} dz' \int_0^{2\pi} d\phi' \overbrace{\cancel{\frac{(z-z')\sin\phi'}{[\ldots]^{3/2}}}}^{0}\hat{y} + \int_{-h/2}^{h/2} dz' \int_0^{2\pi} d\phi' \frac{R}{\left[(z-z')^2 + R^2\right]^{3/2}}\hat{z} \right\}$$

ϕ' integrates to zero

$$= \frac{\mu_0}{4\pi}\sigma\omega R^2 \, R 2\pi \int_{-h/2}^{h/2} \frac{dz'}{\left[(z-z')^2 + R^2\right]^{3/2}}\hat{z}$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0 \sigma \omega R^3}{2} \int_{-h/2}^{h/2} \frac{dz'}{\left[(z-z')^2 + R^2\right]^{3/2}} \hat{z}$$

Let $\quad u = z - z' \quad \Rightarrow \quad du = -dz'$

$$\vec{B} = -\frac{\mu_0 \sigma \omega R^3}{2} \int_{z+h/2}^{z-h/2} \frac{du}{(u^2+R^2)^{3/2}} \hat{z}$$

$$= \frac{\mu_0 \sigma \omega R^3}{2} \int_{z-h/2}^{z+h/2} \frac{du}{(u^2+R^2)^{3/2}} \hat{z}$$

$$= \frac{\mu_0 \sigma \omega R^3}{2} \left. \frac{u}{R^2 (R^2+u^2)^{1/2}} \right|_{z-h/2}^{z+h/2} \hat{z}$$

$$= \frac{\mu_0 \sigma \omega R}{2} \left\{ \frac{z+h/2}{\sqrt{(z+h/2)^2 + R^2}} - \frac{z-h/2}{\sqrt{(z-h/2)^2+R^2}} \right\} \hat{z}$$