Phys 311 2020

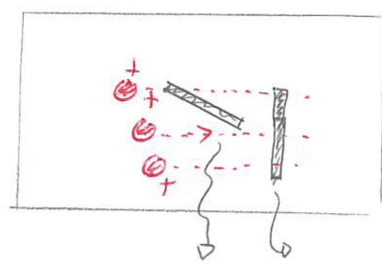
Lecture 27

Fri: HW

Fri: Read 5.2.1, 5.2.2

Two dimensional currents

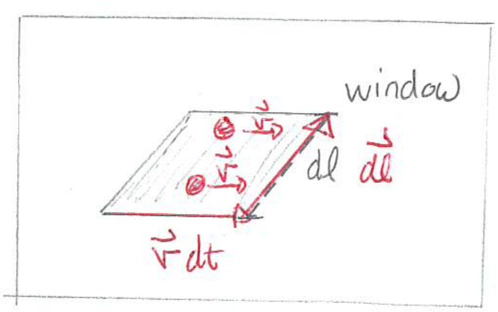
Charge can flow across a two-dimensional surface. Again the basic notion of current is the rate at which charge accumulates. This would refer to some observation "window" and there is a much greater range of choices for these in two dimensions than in one dimension. We have to build this choice into the description of current. The end result will be:



Same size windows, different orientation $\Rightarrow$ different $\left|\frac{dq}{dt}\right|$

1) a definition of a two dimensional surface current density vector
2) a method for using this vector to determine current flowing through any window.

We do this by first considering an infinitesimally small window and charges flowing through this with uniform velocity. Let dl be the length of the window. Then suppose that in time dt, all of the charge that passes through the window is in the shaded region. The charge that passes is



$$dq = \sigma\, da$$

where σ is the charge density and da the area of the parallelogram

Then

$$da = |\vec{v}\,dt \times d\vec{l}\,|$$

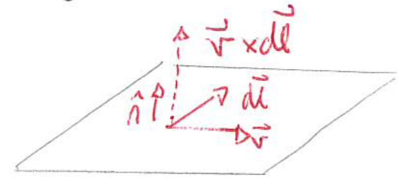
$$\Rightarrow \quad dq = \sigma\,|\vec{v} \times d\vec{l}\,|\,dt \quad \Rightarrow \quad \frac{dq}{dt} = \sigma\,|\vec{v} \times d\vec{l}\,|$$

So the current is

$$I = \sigma\,|\vec{v} \times d\vec{l}\,|$$

We would like to express this without the magnitude sign. Note that $\vec{v} \times d\vec{l}$ is perpendicular to the surface

Let $\hat{n}$ be the indicated normal vector. Then



$$|\vec{v} \times d\vec{l}\,| = (\vec{v} \times d\vec{l}\,) \cdot \hat{n}$$

A standard piece of vector algebra is $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B}$. Thus

$$|\vec{v} \times d\vec{l}\,| = (\hat{n} \times \vec{v}) \cdot d\vec{l}$$

and

$$I = \sigma\,(\hat{n} \times \vec{v}) \cdot d\vec{l}$$

We define the surface current density as

$$\boxed{\vec{K} = \sigma \vec{v}}$$

Units $C/m^2 \; m/s = \frac{C}{ms} = \frac{A}{m}$

Then for this infinitesimal window

$$I = (\hat{n} \times \vec{K}) \cdot d\vec{l} = (d\vec{l} \times \hat{n}) \cdot \vec{K} = (\vec{K} \times d\vec{l}\,) \cdot \hat{n}$$

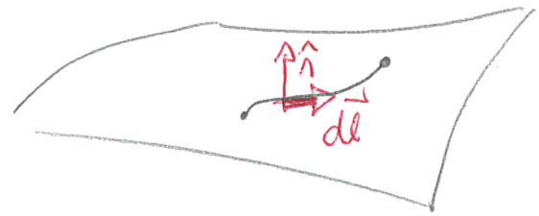
For non-uniform currents and extended windows

$$I = \int_{\text{window}} (\vec{K} \times d\vec{l}\,) \cdot \hat{n} = \int_{\text{window}} (\hat{n} \times \vec{K}) \cdot d\vec{l} = \int_{\text{window}} (d\vec{l} \times \hat{n}) \cdot \vec{K}$$

So the procedure is:

Given a current density $\vec{K}$:

1) specify a window by a line and line elements $\vec{dl}$
2) choose a normal $\hat{n}$
3) then $\vec{dl} \times \hat{n}$ indicates direction of positive current flow or charge accumulation.



4) Current across window is

$$I = \frac{dq}{dt}$$

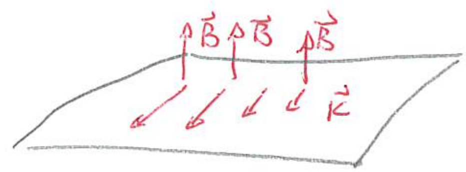
where $q(t)$ is the total charge accumulation in the sense of $\vec{dl} \times \hat{n}$

and

$$I = \int_{\text{window/line}} (\vec{n} \times \vec{K}) \cdot \vec{dl}$$

Force on surface currents

Now consider a sheet of current placed in an external magnetic field. By considering infinitesimal elements we show:

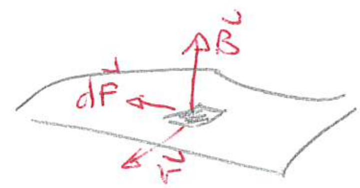


A magnetic field $\vec{B}$ exerts a force on a sheet given by

$$\vec{F}_{mag} = \int_{sheet} \vec{K} \times \vec{B} \, da$$

where $\vec{K}$ is the surface charge density in the sheet.

Proof. Consider an infinitesimal patch with area da. This contains charge dq and the force exerted on it is



$$d\vec{F}_{mag} = dq\, \vec{v} \times \vec{B}$$
$$= \sigma\, da\, \vec{v} \times \vec{B} = \sigma \vec{v} \times \vec{B} \, da$$
$$= \vec{K} \times \vec{B} \, da.$$

Then

$$\vec{F}_{mag} = \int_{sheet} \vec{K} \times \vec{B} \, da \qquad \blacksquare$$

1 Force on a two-dimensional sheet of current.

A surface in the xy plane with $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 1$ carries current with surface current density

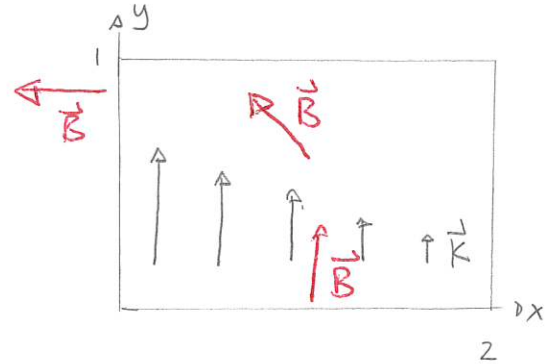
$$\mathbf{K} = \alpha \cos(\pi x/4)\hat{\mathbf{y}}$$

where α is a constant with units of A/m. This is placed in magnetic field

$$\mathbf{B} = \beta\,(x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$$

where β has units of T/m. Determine the force on this sheet of current.

Answer: $\vec{F} = \int \vec{K} \times \vec{B}\, da$



Then $0 \leq x \leq 2$

$0 \leq y \leq 1$

$da = dx\,dy$

$$\vec{K} \times \vec{B} = \alpha \cos\left(\frac{\pi x}{4}\right)\hat{y} \times \left[\beta(x\hat{y} - y\hat{x})\right]$$

$$= \alpha\beta\cos\left(\frac{\pi x}{4}\right)\left[x \underbrace{\cancel{\hat{y} \times \hat{y}}}_{0} - y \underbrace{\cancel{\hat{y} \times \hat{x}}}_{-\hat{z}}\right] = \alpha\beta\cos\left(\frac{\pi x}{4}\right) y\,\hat{z}$$

So

$$\vec{F} = \alpha\beta \int_0^2 dx \int_0^1 dy\; y \cos\left(\frac{\pi x}{4}\right)\hat{z}$$

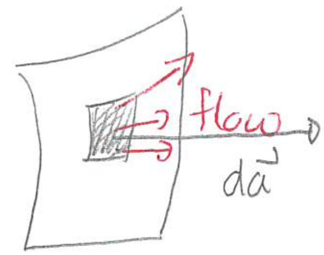
$$= \alpha\beta \underbrace{\int_0^2 \cos\left(\frac{\pi x}{4}\right) dx}_{\frac{4}{\pi}\sin\left(\frac{\pi}{4}x\right)\Big|_0^2 \;=\; \frac{4}{\pi}} \underbrace{\int_0^1 y\,dy}_{1/2}\;\hat{z} = \alpha\beta\,\frac{4}{\pi}\,\frac{1}{2}\,\hat{z}$$

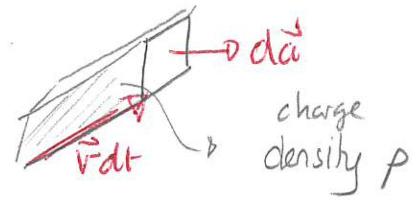
$$\Rightarrow \vec{F} = \frac{2\alpha\beta}{\pi}\hat{z}$$

Currents in three dimensions

In the most general case currents can flow in three dimensions. We can define the current by constructing a surface and determining the rate at which charge flows through the surface.

We can first consider an infinitesimal flat surface with area $d\vec{a}$ and assume charge flow is uniform. The same prior parallelogram region argument gives that the charge flowing is





$$dq = \rho \, \vec{v} dt \cdot d\vec{a}$$

$$\Rightarrow \quad \frac{dq}{dt} = \rho \vec{v} \cdot d\vec{a}$$

$$\Rightarrow \quad I = \rho \vec{v} \cdot d\vec{a}$$

We define the volume current density (current per unit area) as

$$\boxed{\vec{J} = \rho \vec{v}}$$

units A/m^2

and the current passing through this surface is

$$\boxed{I = \int_{\text{entire surface}} \vec{J} \cdot d\vec{a}}$$

A derivation analogous to the two dimensional case gives: The force acting on a volume current is

$$\boxed{\vec{F}_{mag} = \int_{\text{region}} \vec{J} \times \vec{B} \, d\tau}$$

Magnetic Fields Produced by Currents

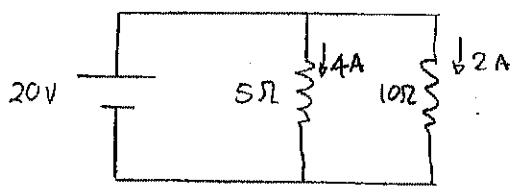
In general observations and experiments indicate

Moving charges and currents produce magnetic fields

We aim to find basic rules by which currents produce fields. We consider two situations:

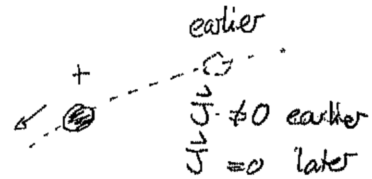
Stationary Currents

The current density at any location does not vary with time although it may be different at different locations



Non-stationary currents

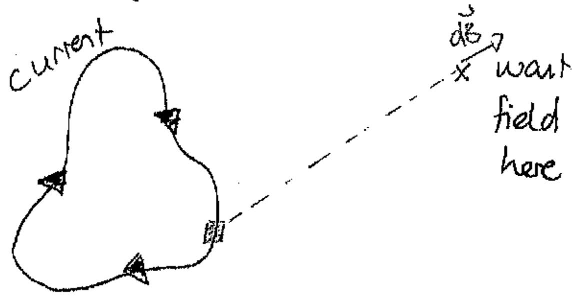
At various locations the current density varies as time passes



We start with stationary currents.

Biot-Savart Law

The general situation is illustrated.



We compute the field by considering contributions from small segments. We will present a rule for the field produced by each segment $d\vec{B}$. Then the total field is

$$\vec{B} = \int_{\text{all loop}} d\vec{B}$$