℞ Tues: HW 16

Weds: 5.1.3. → 5.2.1

Forces on Currents

The basic force law in electromagnetism, the Lorentz Force law, gives the force exerted on a moving point charge particle

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

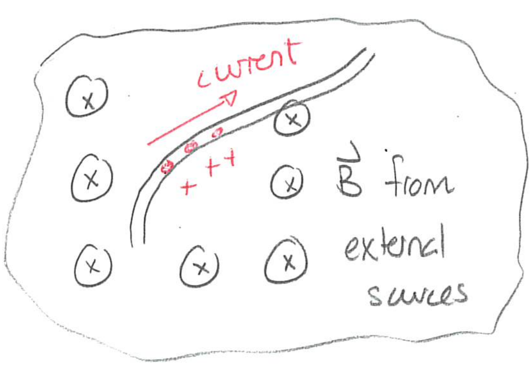
where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields acting on the particle.

We can extend this law to describe situations where electric and magnetic fields act on multiple moving charged particles. This requires us to:

1) describe the collection of moving charged particles via a current

2) describe the force exerted by a field on a current.

We will consider situations where there is no electric field. Then

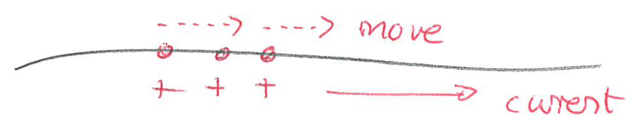


What force does magnetic field $\vec{B}$ exert on the entire current?

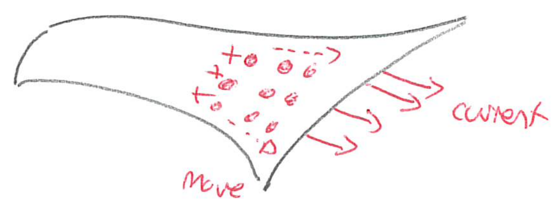
Currents.

A current is an assembly of moving charged particles. We usually describe currents in terms of continuous fluid flows rather than motion of discrete charges. There are three situations:

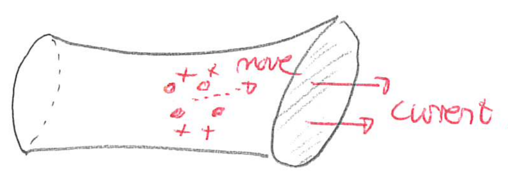
1) one dimensional currents
   (e.g. wire)



2) two dimensional surface currents
   (along a sheet)



3) three dimensional volume currents
   (e.g. in a tube)



Each situation will require a slightly different mathematical description. In all cases the basic concept is:

$\boxed{\text{Current quantifies the rate at which charge flows with time}}$

In general such currents can depend on:

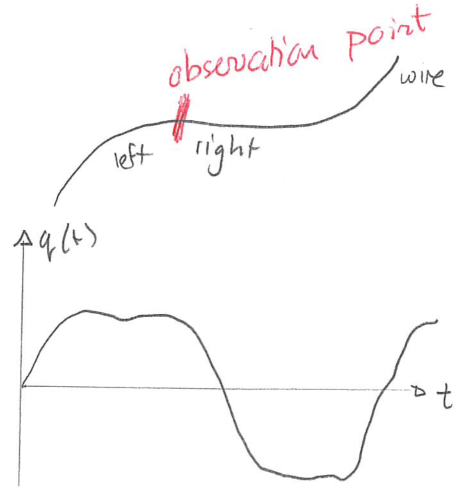
1) physical location (varies from one place to another)
2) time of observation (varies " " moment to ")

One can even have dependence on both.

One dimensional currents

An example of a one dimensional current is a flow of charge along a one dimensional wire. The current will be described by a vector $\vec{I}$, which, in principle, could vary from one location to another and also from one moment to another. The vector is constructed via:

a) magnitude of current



Pick a single observation point and record the total charge that has passed since $t=0$ at that point. Denote this $q(t)$. Then the magnitude of the current is

$$I = \left|\frac{dq}{dt}\right|$$

units: Amperes: $A = C/s$

b) direction of current

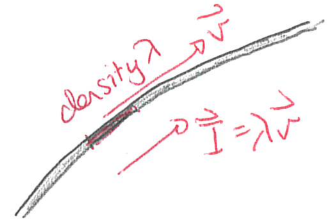
This depends on the sign of the moving charges. In general the direction of the current is:

i) tangent to the wire

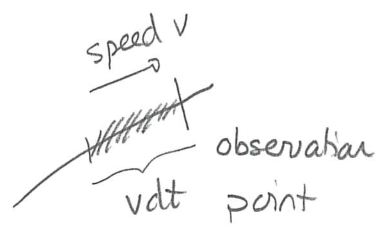
ii) in direction of increasing positive charge accumulation

If current arises from the motion of a continuous linear charge then one can relate the current vector to the linear charge density and the velocity of the moving charges. Specifically:

> If the linear charge density of moving charges is λ and their velocity is $\vec{v}$ then the current is
>
> $$\vec{I} = \lambda \vec{v}$$



<u>Proof:</u> Consider observing charges moving past an observation point.



We can consider all the charge that passes in time dt. This occupies length $dl = v\,dt$ behind the observation point. Thus the charge that flows is $dq = \lambda dl$.

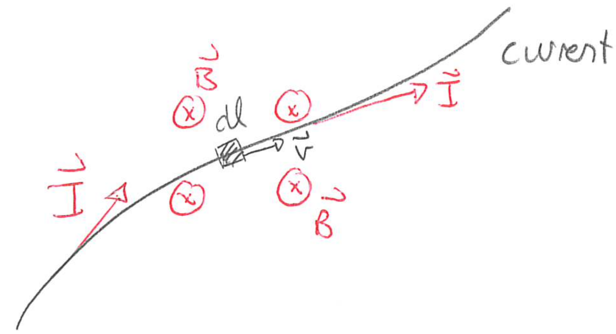
$$\Rightarrow \quad dq = \lambda v dt \quad \Rightarrow \quad \frac{dq}{dt} = \lambda v.$$

This establishes the rule for the magnitude. The direction follows immediately from the velocity $\blacksquare$

Note that both λ and $\vec{v}$ could vary from one location to another and also as time passes. A stationary current is one for which $\vec{I}$ stays constant as time passes (at each single location)

Force exerted by a magnetic field on a one dimensional current

Since a magnetic field exerts a force on a moving charged particle, a magnetic field will also exert a force on a current.



The basic rule can be applied to a segment of length dl.

Suppose the velocity of the segment is $\vec{v}$. Then this is approximately a point charge with charge $dq = \lambda dl$. Then the force on this segment is

$$d\vec{F} = dq\, \vec{v} \times \vec{B}$$

$$= dl\, \lambda \vec{v} \times \vec{B}$$

$$= dl\, \vec{I} \times \vec{B} \quad = \vec{I} \times \vec{B}\, dl$$

Then over the entire current

$$\boxed{\vec{F} = \int_{\text{current}} \vec{I} \times \vec{B}\, dl}$$

An alternative expression uses $\vec{I} dl = I d\vec{l}$ and then

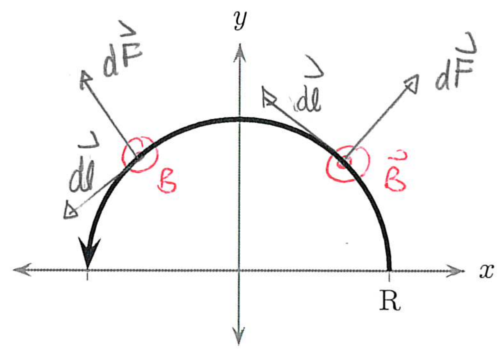
$$\boxed{\vec{F} = \int_{\text{current}} I\, d\vec{l} \times \vec{B}}$$

1 Force on a one-dimensional current.

A semicircular wire with radius R is placed in a magnetic field (in cylindrical coordinates),

$$\mathbf{B} = B \sin\phi\, \hat{\mathbf{z}}.$$

The wire carries current with a uniform magnitude. Determine the net force exerted by the field on the wire.



Answer: By symmetry the force will be along $\hat{y}$.

$$\vec{F} = \int I\, d\vec{l} \times \vec{B} = I \int d\vec{l} \times \vec{B}$$

We need $d\vec{l}$. Here

$$\left.\begin{array}{c} s = R \\ 0 \leq \phi \leq \pi \end{array}\right\} \quad d\vec{l} = R\, d\phi\, \hat{\phi}$$

So

$$d\vec{l} \times \vec{B} = R\, d\phi\, \hat{\phi} \times B \sin\phi\, \hat{z}$$

$$= RB \sin\phi\, d\phi\, \hat{\phi} \times \hat{z} = RB \sin\phi\, d\phi\, \hat{s}$$

Now $\hat{s}$ varies along the curve. So

$$\hat{s} = \cos\phi\, \hat{x} + \sin\phi\, \hat{y}$$

$$\Rightarrow d\vec{l} \times \vec{B} = RB \sin\phi \left(\cos\phi\, \hat{x} + \sin\phi\, \hat{y}\right) d\phi$$

$$= \left(RB \sin\phi \cos\phi\, \hat{x} + RB \sin^2\phi\, \hat{y}\right) d\phi$$

So

$$\vec{F} = I \int_0^{\pi} RB \cos\phi \sin\phi \, d\phi \, \hat{x} + I \int_0^{\pi} RB \sin^2\phi \, d\phi \, \hat{y}$$

$$= IRB \underbrace{\int_0^{\pi} \cos\phi \sin\phi \, d\phi}_{0} \, \hat{x} + IRB \underbrace{\int_0^{\pi} \sin^2\phi \, d\phi}_{\frac{\pi}{2}} \, \hat{y}$$

$$\Rightarrow \quad \vec{F} = IRB \frac{\pi}{2} \hat{y}$$