Phys 311 2020

Lecture 25

Fri: HW 16

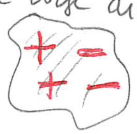
Mon: Read 5.1.3

Tues:

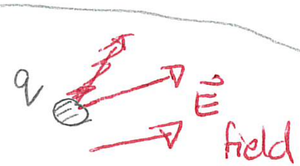
Magnetostatics

Electrostatics describes the forces that stationary source charges exert of test charges. It does so via electric fields.

charge distrib



density ρ



$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

$\vec{\nabla}\times\vec{E} = 0$

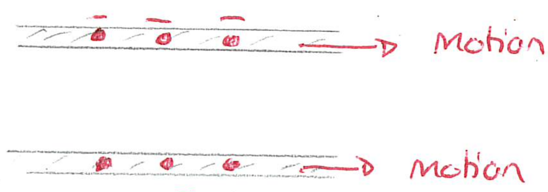
$\rightarrow$ Force on test charge

$\vec{F} = q\vec{E}$

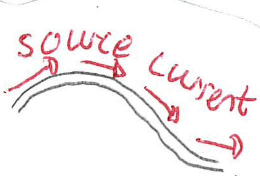
where q = charge of test charge

This does not describe forces exerted by moving charges. Consider two parallel identical currents. We have

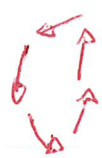
* Coulomb's law predicts $\Rightarrow$ repel
* Observation $\Rightarrow$ attract.



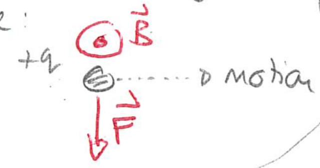
There is clearly a different process here. We will describe this via magnetic fields:



$\rightarrow$ Produces magnetic field $\vec{B}$



$\rightarrow$ Magnetic field exerts force on moving test charge:



There are two parts to this:

1) describe how a magnetic field exerts a force on a moving test charge

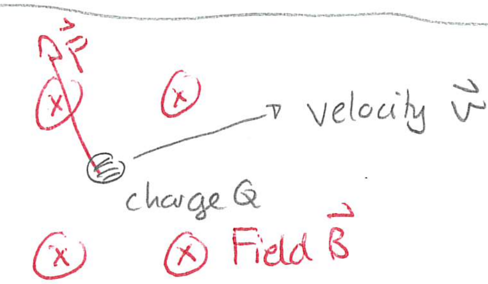
2) describe how currents produce magnetic fields.

Force exerted by a magnetic field

We will assume that there is a mechanism for describing how magnetic fields are produced. An indirect definition involves the force exerted on a moving charged particle. Observations eventually lead to:

If a particle with charge Q moves with velocity $\vec{v}$ through a magnetic field $\vec{B}$ then the force exerted by the field on the charge is

$$\vec{F} = Q\vec{v} \times \vec{B}$$



Note that:

1) the force produced by a magnetic field is perpendicular to the field

2) the force tends to produce changes in the direction of motion

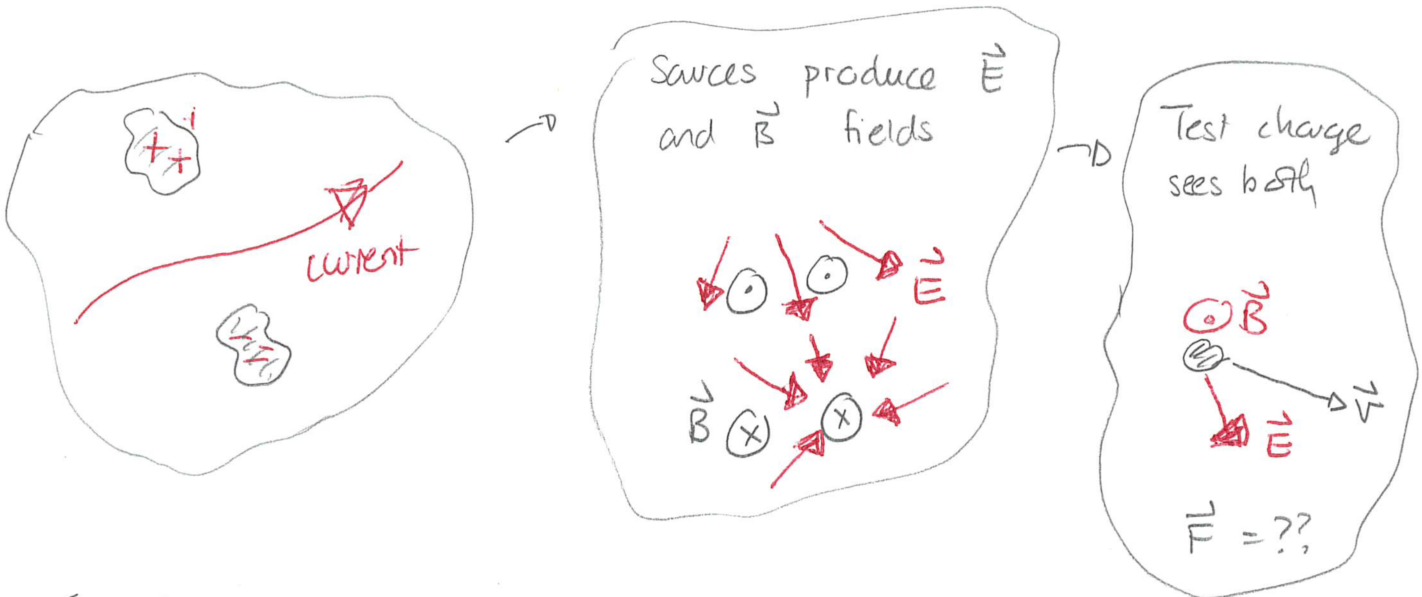
3) the units satisfy

$$N = C\ m/s \times \text{units of } B$$

$$\Rightarrow \text{units of } B = \frac{Ns}{C \cdot m} = \frac{N}{A \cdot m} \equiv \text{Tesla} \equiv T$$

Lorentz Force Law

Suppose that a test particle is subjected to both electric and magnetic fields. These each exert a force on the particle and the two forces can be combined.



The fundamental rule that describes this is the Lorentz Force Law

Suppose a test particle with charge Q is in an electric and magnetic field. Then the force exerted on the test charge is

$$\vec{F} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

where $\vec{v}$ = velocity of particle

$\vec{E}$ = electric field at particle location

$\vec{B}$ = magnetic " " " "

This is the fundamental force law in electromagnetic theory. Then Newton's 2nd Law, $\vec{F}_{net} = m\vec{a}$ determines acceleration

1 Particle in a uniform magnetic field

A particle with mass m and charge Q is placed in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ (here B is independent of location and time). The electric field in this region is $\mathbf{E} = 0$. We aim to convert the equations of motion for this particle into a set of differential equations and to then solve these, knowing the initial velocity.

a) Let $\mathbf{v} = v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}} + v_z(t)\hat{\mathbf{z}}$ be the velocity of the particle. Combine Newton's second law and the Lorentz force law to produce a set of coupled differential equations for $v_x(t), v_y(t)$ and $v_z(t)$. Rewrite these in terms of the cyclotron frequency

$$\omega := \frac{QB}{m}.$$

b) Solve the differential equation for $v_z(t)$ in terms of the initial velocity $v_z(0)$.

c) Manipulate the two first order differential equations for v_x and v_y to produced uncoupled second order differential equations for each of these.

d) Solve each of these to yield $v_x(t)$ and $v_y(t)$ in terms of $v_x(0)$ and $v_y(0)$.

e) Use these to find expressions for the components of the position of the particle, $x(t), y(t)$ and $z(t)$ (in terms of initial positions and velocities).

f) Describe the trajectory followed by the particle geometrically. What does the cyclotron frequency describe in terms of the particle motion?

<u>Answer:</u> a) $\vec{F}_{net} = m\vec{a}$

$$\Rightarrow \quad m\frac{d\vec{v}}{dt} = \vec{F}_{net}$$

$$\Rightarrow \quad m\frac{d\vec{v}}{dt} = Q\,\vec{v}\times\vec{B}$$

Now $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

$$\vec{B} = B\hat{z}$$

$$\Rightarrow \quad \vec{v}\times\vec{B} = v_xB\underbrace{\hat{x}\times\hat{z}}_{-\hat{y}} + v_yB\underbrace{\hat{y}\times\hat{z}}_{\hat{x}} + v_zB\cancel{\hat{z}\times\hat{z}}^{\,0}$$

So $$Q\,\vec{v}\times\vec{B} = Qv_yB\,\hat{x} - Qv_xB\,\hat{y}$$

But $$\frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{x} + \frac{dv_y}{dt}\hat{y} + \frac{dv_z}{dt}\hat{z}$$

Thus $$m\frac{d\vec{v}}{dt} = m\frac{dv_x}{dt}\hat{x} + m\frac{dv_y}{dt}\hat{y} + m\frac{dv_z}{dt}\hat{z}$$

$$= Qv_yB\,\hat{x} - Qv_xB\,\hat{y}$$

Comparing coefficients gives:

$$\left.\begin{aligned} m\frac{dv_x}{dt} &= QB\,v_y \\ m\frac{dv_y}{dt} &= -QB\,v_x \\ m\frac{dv_z}{dt} &= 0 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \frac{dv_x}{dt} &= \omega v_y \\ \frac{dv_y}{dt} &= -\omega v_x \\ \frac{dv_z}{dt} &= 0 \end{aligned}}$$

b) $v_z(t) = \text{constant}$ $\Rightarrow$ $v_z(t) = v_z(0) \equiv \text{constant}$

c) $$\frac{d}{dt}\frac{dv_x}{dt} = \omega\frac{dv_y}{dt}$$

$$\Rightarrow \frac{d^2v_x}{dt^2} = \omega(-\omega v_x) \Rightarrow \frac{d^2v_x}{dt^2} = -\omega^2 v_x$$

$$\frac{d}{dt}\left(\frac{dv_y}{dt}\right) = -\omega\frac{dv_x}{dt} \Rightarrow \frac{d^2v_y}{dt^2} = -\omega^2 v_y$$

d) Consider

$$\frac{d^2v_x}{dt^2} = -\omega^2 v_x$$

The general solution is

$$v_x(t) = A\cos\omega t + B\sin\omega t$$

where A, B are constants

We could attempt to construct a similar solution for v_y independently. But v_x, v_y are related. For example

$$\frac{dv_x}{dt} = \omega v_y \Rightarrow v_y = \frac{1}{\omega}\frac{dv_x}{dt}$$

Thus $$v_y(t) = \frac{1}{\omega}\left[-A\omega \sin \omega t + B\omega \cos \omega t\right]$$

$$= -A \sin \omega t + B \cos \omega t$$

So we have a general solution:

$$v_x(t) = A \cos \omega t + B \sin \omega t$$
$$v_y(t) = -A \sin \omega t + B \cos \omega t$$

Then $$v_x(0) = A \underbrace{\cos 0}_{1} + B \cancel{\sin 0}^{0} \Rightarrow A = v_x(0)$$

$$v_y(0) = -A \sin 0 + B \underbrace{\cos 0}_{1} \Rightarrow B = v_y(0)$$

Thus the solution is:

$$v_x(t) = v_x(0) \cos(\omega t) + v_y(0) \sin(\omega t)$$
$$v_y(t) = -v_x(0) \sin(\omega t) + v_y(0) \cos(\omega t)$$
$$v_z(t) = v_z(0)$$

e) We have

$$V_x(t) = \frac{dx}{dt} = V_x(0)\cos(\omega t) + V_y(0)\sin(\omega t)$$

$$V_y(t) = \frac{dy}{dt} = -V_x(0)\sin(\omega t) + V_y(0)\cos(\omega t)$$

$$V_z(t) = \frac{dz}{dt} = V_z(0)$$

Each can be integrated independently. So

$$x(t) = \frac{V_x(0)}{\omega}\sin(\omega t) - \frac{V_y(0)}{\omega}\cos(\omega t) + C_x$$ constant

$$y(t) = \frac{V_x(0)}{\omega}\cos(\omega t) + \frac{V_y(0)}{\omega}\sin(\omega t) + C_y$$

$$z(t) = V_z(0)\,t + C_z$$

The remaining constants can be fixed in terms of initial locations So

$$x(0) = \frac{V_x(0)}{\omega}\overset{0}{\cancel{\sin(0)}} - \frac{V_y(0)}{\omega}\overset{1}{\cancel{\cos 0}} + C_x$$

$$\Rightarrow \quad C_x = x(0) + \frac{V_y(0)}{\omega}$$

$$y(0) = \frac{V_x(0)}{\omega}\overset{1}{\cancel{\cos 0}} + \frac{V_y(0)}{\omega}\cancel{\sin 0} + C_y$$

$$\Rightarrow \quad C_y = y(0) - \frac{V_x(0)}{\omega}$$

$$z(0) = V_z(0)\,0 + C_z$$

$$\Rightarrow \quad C_z = z(0)$$

We can rewrite these in terms of

$$V_x(0) = V_{ox}$$

$$x(0) = x_0$$

etc...

Thus

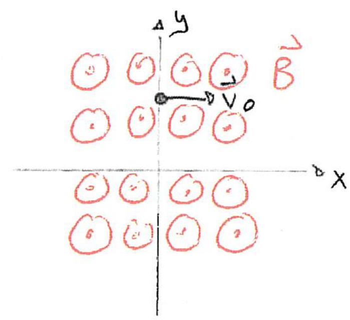
$$\boxed{\begin{aligned} x(t) &= \frac{V_{ox}}{\omega}\sin(\omega t) - \frac{V_{oy}}{\omega}\cos(\omega t) + \frac{V_{oy}}{\omega} + x_0 \\ y(t) &= \frac{V_{ox}}{\omega}\cos(\omega t) + \frac{V_{oy}}{\omega}\sin(\omega t) - \frac{V_{ox}}{\omega} + y_0 \\ z(t) &= V_{oz}t + z_0 \end{aligned}}$$

f) Consider the simplest case:

$V_{oy} = 0$ $\qquad z_0 = 0$

$V_{oz} = 0$

$V_{ox} = V_0 \neq 0$



Then:

$$x(t) = \frac{V_0}{\omega}\sin(\omega t) + x_0$$

$$y(t) = \frac{V_0}{\omega}\cos(\omega t) - \frac{V_0}{\omega} + y_0$$

$$z(t) = 0$$

This is an orbit in the xy plane since $z=0$

Then

$$x(t) - x_0 = \frac{V_0}{\omega} \sin(\omega t)$$

$$y(t) - y_0 + \frac{V_0}{\omega} = \frac{V_0}{\omega} \cos(\omega t)$$

Let $\alpha = x_0$

$\beta = y_0 - \frac{V_0}{\omega}$

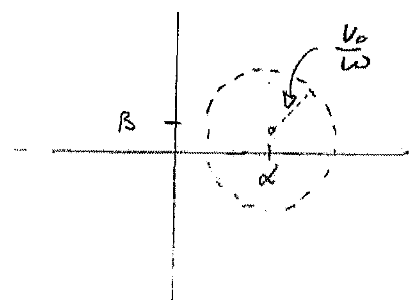
Then $x(t) - \alpha = \frac{V_0}{\omega} \sin(\omega t)$

$y(t) - \beta = \frac{V_0}{\omega} \cos(\omega t)$

So $(x(t)-\alpha)^2 + (y(t)-\beta)^2 = \left(\frac{V_0}{\omega}\right)^2$. This describes a circle with radius $\frac{V_0}{\omega}$ centered at (α, β).

The orbit is counterclockwise with angular frequency

$$\omega = \frac{QB}{m}$$



Such circular motion is common in uniform magnetic fields.
The motion is more complicated if there is a component of velocity along the field direction. In this case it is helical

Demo: PSU - S

Charge in Unif field I

Charge in unif field II

Charge in bottle etc...