Phys 311 2020

Lecture 24

Fri: 5.1.1, 5.1.2.

HW 15

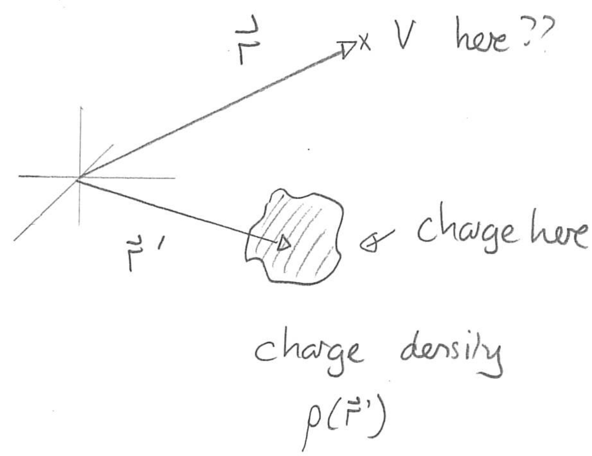
General multipole expansion

The general idea with the multipole expansion is to approximate the potential produced by a localized distribution. We found that

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}')\,d\tau'$$

$$+\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}')\frac{\vec{r}'\cdot\vec{r}}{r^2}\,d\tau'$$

$$+\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}')\frac{1}{2}\left[\frac{3(\vec{r}'\cdot\vec{r})^2}{r^4}-\left(\frac{r'}{r}\right)^2\right]d\tau'$$



Note that $\vec{r}/r = \hat{r}$. Then:

$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}')\,d\tau' \quad \leftarrow \text{monopole} \\ &+ \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int \rho(\vec{r}')\,\vec{r}'\cdot\hat{r}\,d\tau' \quad \leftarrow \text{dipole} \\ &+ \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int \rho(\vec{r}')\frac{1}{2}\left[3(\vec{r}'\cdot\hat{r})^2 - r'^2\right]d\tau' + \dots \quad \leftarrow \text{quadrupole} \end{aligned}}$$

Note that:

1) each integral only involves
   - the charge distribution
   - the unit vector $\hat{r}$ in field direction.

2) the terms contain increasing powers of $(\frac{1}{r})$ and for large distances these diminish as the series continues.

3) the expansion will be useful when $r \gg r'$ for all $\vec{r}'$ in the distribution.

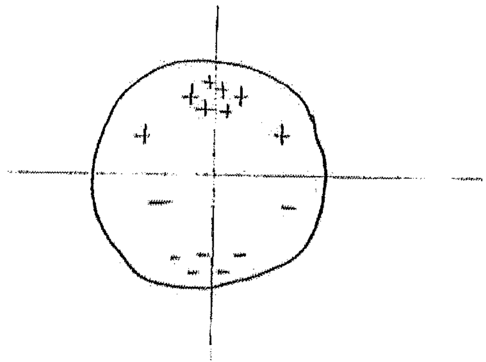
1 Spherical charge distribution: monopole term

Charge is distributed within a sphere of radius R according to

$$\rho(\mathbf{r}') = \frac{3\alpha}{4\pi}\frac{\cos\theta'}{R^3}$$

where α has units of charge. Sketch the charge distribution qualitatively and determine the monopole contribution to the potential.

Answer:



The monopole term requires

$$\int \rho(\vec{r}')\,d\tau'$$

and here

$$\left.\begin{array}{l} 0 \le r' \le R \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\} \quad d\tau' = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$$

So

$$\int \rho(\vec{r}')\,d\tau' = \frac{3\alpha}{4\pi R^3}\int_0^R dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta'\, r'^2 \sin\theta' \cos\theta'$$

$$= \frac{3\alpha}{4\pi R^3}\int_0^R r'^2\, dr' \int_0^{2\pi} d\phi' \underbrace{\int_0^{\pi} d\theta' \sin\theta' \cos\theta'}_{\frac{1}{2}\sin^2\theta' \big|_0^{\pi} = 0}$$

$\Rightarrow$ monopole contribution is zero.

Now consider the various terms.

Monopole term

Here

$$V_{mon} = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_{\text{all space}} \rho(\vec{r}\,')\,d\tau'$$

and the integral is simply the total charge in the distribution

$$\boxed{Q = \int_{\text{all space}} \rho(\vec{r}\,')\,d\tau'}$$

This is also called the monopole moment. Then

$$\boxed{V_{mon} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}}$$

Thus the monopole contribution is exactly that of a point charge at the origin where the charge equals the total charge in the distribution

Dipole term

The dipole term is

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_{\text{all space}} \rho(\vec{r}\,')\underbrace{\vec{r}\,'\cdot\hat{r}}_{\hat{r}\cdot\vec{r}\,'}\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{r}\cdot\underbrace{\left[\int_{\text{all space}}\rho(\vec{r}\,')\,\vec{r}\,'\,d\tau'\right]}$$

only depends on charge distribution

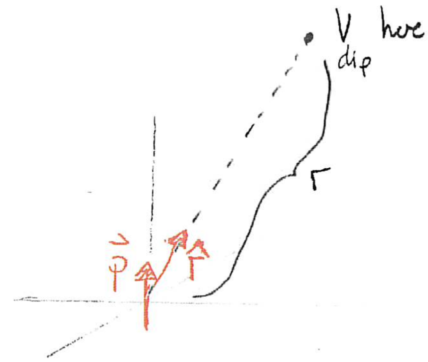
We define the electric dipole moment as:

$$\boxed{\vec{p} := \int_{\text{all space}} \rho(\vec{r}\,')\,\vec{r}\,'\,d\tau'}$$

This:

1) is a vector
2) has units Cm

Then the dipole term is:



$$\boxed{V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}}$$

For other distributions:

Surface charge	$\vec{p} =$	$\int \sigma(\vec{r}\,')\,\vec{r}\,'\,da'$
Line charge	$\vec{p} =$	$\int \lambda(\vec{r}\,')\,\vec{r}\,'\,dl'$
Point charges	$\vec{p} =$	$\sum q_i \vec{r}_i\,'$

2 Spherical charge distribution: dipole term

Charge is distributed within a sphere of radius R according to

$$\rho(\mathbf{r}') = \frac{3\alpha}{4\pi}\frac{\cos\theta'}{R^3}$$

where α has units of charge. Determine the dipole moment and the dipole contribution to the potential.

Answer:

$$\vec{p} = \int \rho(\vec{r}')\,\vec{r}'\,d\tau'$$

Here $d\tau' = r'^2 \sin\theta'\,dr'\,d\theta'\,d\phi'$ $\qquad 0 \leq r' \leq R$

$0 \leq \theta' \leq \pi$

$0 \leq \phi' \leq 2\pi$

Also $\vec{r}' = r'\cos\phi'\sin\theta'\,\hat{x} + r'\sin\phi'\sin\theta'\,\hat{y} + r'\cos\theta'\,\hat{z}$

$$\vec{p} = \frac{3\alpha}{4\pi R^3}\left\{\int_0^R dr'\int_0^\pi d\theta'\int_0^{2\pi} d\phi'\, r'^2\sin\theta'\; r'\underbrace{\cos\phi'}_{\text{integrates to } 0}\sin\theta'\,\cos\theta'\,\hat{x}\right.$$

$$+ \int\cdots\int\cdots\int r'^2\sin\theta'\; r'\overbrace{\sin\phi'}\sin\theta'\,\cos\theta'\,\hat{y}$$

$$\left. + \int\int\int r'^2\sin\theta'\; r'\cos\theta'\,\cos\theta'\,\hat{z}\right\}$$

$$= \frac{3\alpha}{4\pi R^3}\underbrace{\int_0^R r'^3\,dr'}_{\frac{1}{4}R^4}\;\underbrace{\int_0^\pi \sin\theta'\cos^2\theta'}_{-\frac{1}{3}\cos^3\theta'\big|_0^\pi}\;\underbrace{\int_0^{2\pi} d\phi'}_{2\pi}\;\hat{z}$$

$$= \frac{\cancel{3}\alpha}{\cancel{2}\cancel{4}\pi R^3}\,\frac{R^4}{4}\,\frac{\cancel{2}}{\cancel{3}}\,\cancel{2}\cancel{\pi}\,\hat{z} \quad \Rightarrow \quad \vec{p} = \frac{\alpha R}{4}\,\hat{z}$$

So

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \vec{p} \cdot \hat{r}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\alpha R}{4} \underbrace{\hat{z} \cdot \hat{r}}_{\cos\theta}$$

$$V_{dip} = \frac{\alpha R}{16\pi\epsilon_0 r^2} \cos\theta$$

3 Monopole and dipole moments for a surface charge distribution

Charge on a disk of radius R lying in the xy plane distributed according to

$$\sigma(\mathbf{r}') = \frac{q}{2R^2} \sin\left(\frac{\phi'}{2}\right).$$

a) Determine the monopole moment.

b) Determine the dipole moment.

c) Determine the monopole and dipole terms in the electrostatic potential.

Answer: a)

$$Q = \int \rho(\vec{r}\,')\,da'$$

$$0 \le r' \le R$$

$$0 \le \phi' \le 2\pi$$

$$= \frac{q}{2R^2}\int_0^R dr' \int_0^{2\pi} d\phi'\, r' \sin\frac{\phi'}{2} \qquad da' = r'dr'd\phi'$$

$$= \frac{q}{2R^2}\underbrace{\int_0^R r'dr'}_{R^2/2} \underbrace{\int_0^{2\pi} \sin\phi'/2\, d\phi'}_{\underbrace{-2\cos(\phi'/2)\big|_0^{2\pi}}_{4}}$$

$$\Rightarrow \quad Q = q$$

b) $$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,da' \qquad \vec{r}\,' = r'\hat{r}$$

$$= r'\cos\phi'\hat{x} + r'\sin\phi'\hat{y}$$

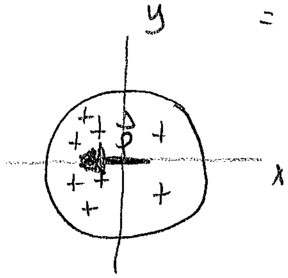
$$= \int_0^R dr' \int_0^{2\pi} d\phi' r'\left[r'\cos\phi'\hat{x} + r'\sin\phi'\hat{y}\right]\frac{q}{2R^2}\sin\left(\frac{\phi'}{2}\right)$$

$$= \frac{q}{2R^2}\underbrace{\int_0^R r'^2 dr'}_{R^3/3} \int_0^{2\pi} \sin\frac{\phi'}{2}\left\{\left[2\cos^2\frac{\phi'}{2} - 1\right]\hat{x} + 2\sin\frac{\phi'}{2}\cos\frac{\phi'}{2}\hat{y}\right\}$$

$$\vec{p} = \frac{q}{2R^2}\,\frac{R^3}{3}\left[\left(-\frac{2}{3}\cdot 2\cos^3\frac{\phi'}{2} + 2\cos\frac{\phi'}{2}\right)\hat{x} + \cancel{\frac{2}{3}\,2\sin^3\frac{\phi'}{2}\,\hat{y}}\right]_0^{2\pi}$$

$$= \frac{qR}{6}\left[\left(-\frac{4}{3}(-1) - 2\right)2\,\hat{x}\right]$$

$$= \frac{qR}{6}\left(-\frac{2}{3}\cdot 2\right)\hat{x} \quad \Rightarrow \quad \vec{p} = -\frac{4}{18}\,qR\,\hat{x}$$



$$\vec{p} = -\frac{2}{9}\,qR\,\hat{x}$$

c) Then $V_{mon} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r} \Rightarrow V_{mon} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r}$

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{r}\cdot\vec{p}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\left(-\frac{2}{9}\,qR\,\hat{r}\cdot\hat{x}\right)$$

$$V_{dip} = -\frac{qR}{18\pi\epsilon_0 r^2}\,\hat{r}\cdot\hat{x}$$

In spherical co-ordinates $\hat{r} = \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z}$

$$\Rightarrow \hat{r}\cdot\hat{x} = \cos\phi\sin\theta$$

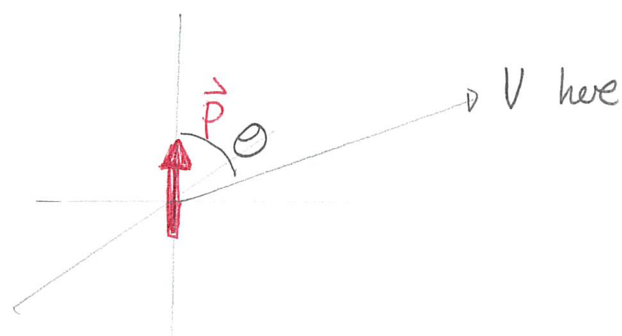
$$\Rightarrow V_{dip} = \frac{-qR}{18\pi\epsilon_0 r^2}\cos\phi\sin\theta$$

Point dipoles

So far we have constructed dipoles from extended charge distributions and have used the resulting dipole moments to compute potentials. We can also consider idealized point dipoles – these would be charged entities with no spatial extent but which have non-zero dipole moments. Approximate examples would be certain atoms and molecules. Generally such dipoles are assumed to have zero monopole moment.

Consider such a dipole oriented along the z-axis. Then the resulting potential is

$$V = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\vec{p}\cdot\hat{r}$$



$$V = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\hat{z}\cdot\hat{r}$$

Using spherical co-ordinates, $\hat{z}\cdot\hat{r} = \cos\theta$. Thus

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta}$$

Demo: PhET charges + fields

–create dipole – draw equipotentials

Electric field produced by a dipole

For any

$$V = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta$$

Then the electric field produced by this is

$$\vec{E} = -\vec{\nabla}V$$

$$= -\left\{ \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial V}{\partial \phi}}\hat{\phi} \right\}$$

$$= -\left\{ -\frac{2p}{4\pi\epsilon_0 r^3}\cos\theta\,\hat{r} - \frac{1}{r}\frac{p}{4\pi\epsilon_0 r^2}\sin\theta\,\hat{\theta} \right\}$$

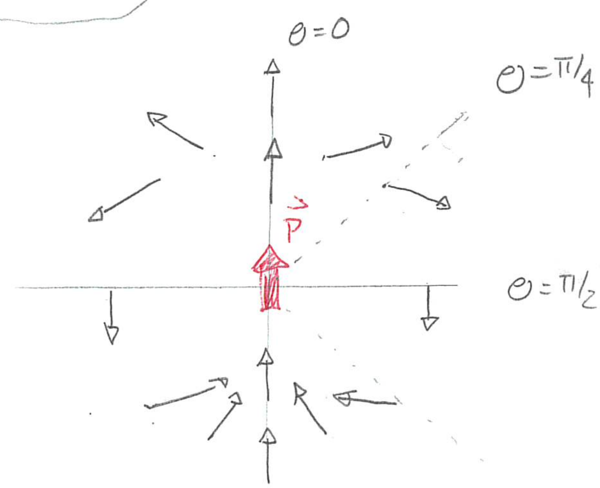
$$= \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}\left\{ 2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} \right\}$$

So

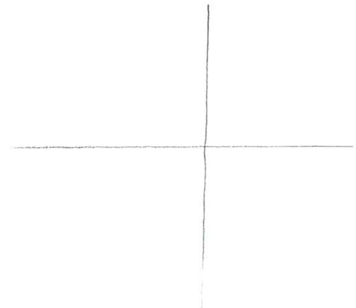
$$\boxed{\vec{E} = \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

Demo: Falstad Dipole Field

The field can be mapped using spherical co-ordinates



We can find locations where the dipole field points radially inward toward the z-axis. At these special locations another charged particle will experience a radially inward force. To find these:



$$\hat{r} = \sin\theta \cos\phi \, \hat{x} + \sin\theta \sin\phi \, \hat{y} + \cos\theta \, \hat{z}$$

$$\hat{\theta} = \cos\theta \cos\phi \, \hat{x} + \cos\theta \sin\phi \, \hat{y} - \sin\theta \, \hat{z}$$

Then:

$$\vec{E} = \frac{P}{4\pi\epsilon_0} \frac{1}{r^3} \Big\{ \left(2\cos\theta \sin\theta \cos\phi + \sin\theta \cos\theta \cos\phi\right) \hat{x} + \left(2\cos\theta \sin\theta \sin\phi + \sin\theta \cos\theta \sin\phi\right) \hat{y} + \left(2\cos^2\theta - \sin^2\theta\right) \hat{z} \Big\}$$

$$= \frac{P}{4\pi\epsilon_0} \frac{1}{r^3} \Big\{ 3\cos\theta \sin\theta \underbrace{\left(\cos\phi \, \hat{x} + \sin\phi \, \hat{y}\right)}_{\text{radially away from } z} + \left(1 - 3\sin^2\theta\right) \hat{z} \Big\}$$

So when $1 - 3\sin^2\theta = 0$ then the electric field points radially. A particle at these locations can then orbit around the z-axis