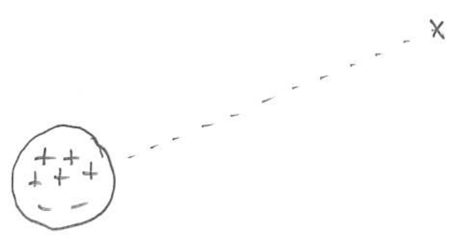
Tues: HW

Weds: 3.4.2. 5.1.1. → 5.1.2.

Multipole expansion

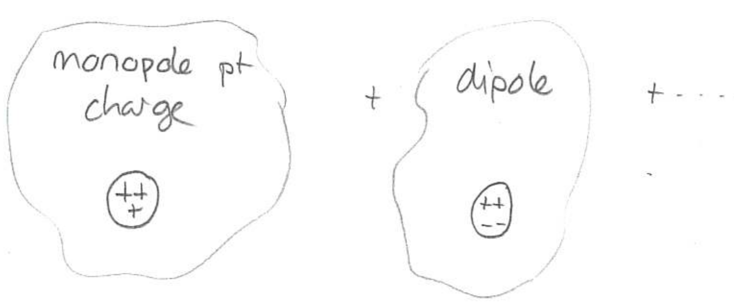
The multipole expansion is an approximation technique which produces a series expression for the potential produced by a localized charge distribution



$$V = V_{monopole} + V_{dipole} + V_{quadrupole}$$

typically diminishing

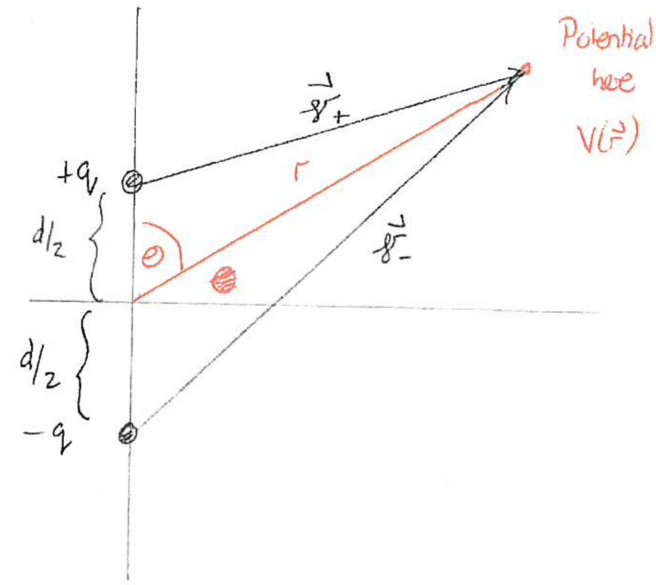
equivalent to



We will generate quantities to describe the monopole and dipole nature of the charge distributions and how these eventually contribute to the potential

Point charge dipole

A point charge dipole consists of two point particles whose charges are exactly opposite. We suppose that they are separated by distance d. Using the illustrated set up we aim to:



Determine exact potential V

$\downarrow$

Determine approximate potential at large distances: dipole potential V_{dipole}

$\downarrow$

Determine approximat dipole field at large distances: $\vec{E}_{dipole} = -\vec{\nabla} V_{dipole}$

Using the illustrated co-ordinate system we aim to find the potential at $\vec{r}$ in terms of r, θ, q, d. Then

$$V(\vec{r}) = V_{from\,+q}(\vec{r}) + V_{from\,-q}(\vec{r})$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r_+} + \frac{1}{4\pi\epsilon_0}\frac{(-q)}{r_-}$$

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{r_+} - \frac{1}{r_-}\right\}$$

Now we aim to consider $r_+, r_- \gg d$. It would initially appear that $r_+ \approx r_-$ and $V \approx 0$. However, we aim to find the deviation from $V = 0$.

1 Point charge dipole

Consider the point charge dipole as set up in the lecture notes.

a) Express $\mathscr{r}_+$ and $\mathscr{r}_-$ in terms of r, d and θ.

b) Substitute into the exact expression for electrostatic potential to show that

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\left[\left(1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-1/2} - \left(1 + \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-1/2}\right]$$

c) Use the Taylor series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3$$

to approximate $V(\mathbf{r})$ to lowest order in d/r.

Answer: a)

$$\vec{r} = \vec{\mathscr{r}}_+ + \frac{d}{2}\hat{z}$$

$$\Rightarrow \quad \vec{\mathscr{r}}_+ = \vec{r} - d/2\,\hat{z}$$

Similarly $\quad \vec{\mathscr{r}}_- = \vec{r} + d/2\,\hat{z}$

Then $\quad \mathscr{r}_+ = \sqrt{\vec{\mathscr{r}}_+ \cdot \vec{\mathscr{r}}_+}$

$$= \left[\left(\vec{r} - \frac{d}{2}\hat{z}\right)\left(\vec{r} - \frac{d}{2}\hat{z}\right)\right]^{1/2}$$

$$= \left[r^2 - 2\,\vec{r}\cdot\frac{d}{2}\hat{z} + \frac{d^2}{4}\right]^{1/2}$$

$$= \left[r^2 - d\,\vec{r}\cdot\hat{z} + d^2/4\right]^{1/2}$$

But $\quad \vec{r}\cdot\hat{z} = r\cos\theta$

$$\Rightarrow \quad \mathscr{r}_+ = \left[r^2 - rd\cos\theta + d^2/4\right]^{1/2}$$

A similar calculation gives:

$$r_- = \left[r^2 + rd\cos\theta + \frac{d^2}{4}\right]^{1/2}$$

b) $$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\left[r^2 - rd\cos\theta + d^2/4\right]^{-1/2} - \left[r^2 + rd\cos\theta + d^2/4\right]^{-1/2}\right]$$

Now $\left[r^2 - rd\cos\theta + d^2/4\right]^{-1/2}$

$$= \left[r^2\left(1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)\right]^{-1/2} = r^{-1}\left[1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right]^{-1/2}$$

Thus

$$V = \frac{q}{4\pi\epsilon_0}\left\{r^{-1}\left[1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right]^{-1/2} - r^{-1}\left[1 + \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right]^{-1/2}\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r}\left\{\left(1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-1/2} - \left(1 + \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-1/2}\right\}$$

c) Here $$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{-\frac{1}{2}(-\frac{3}{2})}{2}x^2 + \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})}{6}x^3$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \ldots$$

Now consider

$$\left(1+\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}\right)^{-1/2}$$

and let $x = +\frac{d}{r}\cos\theta + \frac{d^2}{4r^2}$. If $d \ll r$ then this is small so

$$\left(1+\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}\right)^{-1/2} = 1-\frac{1}{2}\left(+\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}\right)$$
$$+\frac{3}{8}\left(+\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}\right)^2$$
$$-\frac{5}{16}\left(+\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}\right)^3+\cdots$$

$$= 1-\frac{d}{2r}\cos\theta-\frac{d^2}{8r^2}$$
$$+\frac{3}{8}\left(\frac{d^2}{r^2}\cos^2\theta+\frac{d^3}{2r^3}\cos\theta+\frac{d^4}{16r^4}\right)$$
$$-\frac{5}{16}\left(\frac{d^3}{r^3}\cos^3\theta+\frac{3d^4}{4r^4}\cos^2\theta+\frac{3d^5}{16r^5}+\frac{d^6}{64r^6}\right)$$
$$+\cdots$$

$$= 1-\frac{d}{2r}\cos\theta+\frac{d^2}{8r^2}\left(3\cos^2\theta-1\right)$$
$$+\frac{d^3}{16r^3}\cos\theta\left(3-5\cos^2\theta\right)+\cdots$$

Similarly

$$\left(1-\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}\right)^{-1/2} = 1+\frac{d}{2r}\cos\theta+\frac{d^2}{8r^2}\left(3\cos^2\theta-1\right)$$
$$-\frac{d^3}{16r^3}\cos\theta\left(3-5\cos^2\theta\right)$$

So

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\left\{ 1 + \frac{d}{2r}\cos\theta + \frac{d^2}{8r^2}\left(3\cos^2\theta - 1\right) - \frac{d^3}{16r^3}\cos\theta\left(3 - 5\cos^2\theta\right) \right.$$

$$- \left[1 - \frac{d}{2r}\cos\theta + \frac{d^2}{8r^2}\left(3\cos^2\theta - 1\right) + \frac{d^3}{16r^3}\cos\theta\left(3 - 5\cos^2\theta\right)\right]$$

$$\left. + \dots \right\}.$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r}\left\{ 2\frac{d}{2r}\cos\theta - 2\frac{d^3}{16r^3}\cos\theta\left(3 - 5\cos^2\theta\right) + \dots \right\}.$$

$$\Rightarrow V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\left\{ \frac{d}{r}\cos\theta - \frac{d^3}{8r^3}\cos\theta\left(3 - 5\cos^2\theta\right) + \dots \right\}$$

If $r \gg d$ then the first term is much larger than the others. So

$$V(r) \approx \frac{1}{4\pi\epsilon_0}\frac{qd}{r^2}\cos\theta$$

Thus

For a point charge dipole

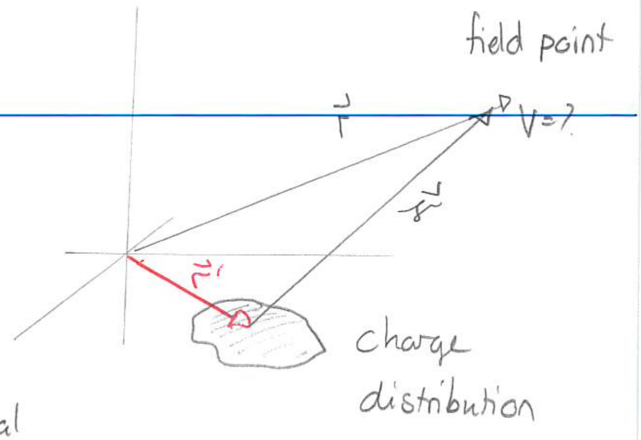
$$V(r) \approx \frac{1}{4\pi\epsilon_0}\frac{qd}{r}\cos\theta$$

where d is the separation between the charges and θ is the illustrated angle

General multipole expansion

Now consider an arbitrary localized charge distribution. Can we extract a monopole, dipole and quadrupole term from this?

Specifically we will want the potential at a location $\vec{r}$ such that $r \gg r'$ for all r' within the distribution.



We will:

1) decompose the distribution into segments
2) determine the contribution to the potential from each segment. This will depend on $\vec{r}, \vec{r}\,'$ and $\vec{\mathscr{r}}$
3) rewrite the expression in terms of ratios of the type r'/r

The illustrated segment contributes:

$$dV = \frac{1}{4\pi\epsilon_0} \frac{\rho(\vec{r}\,')\, d\tau'}{\mathscr{r}}$$

and $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \Rightarrow$

$$\begin{aligned}
\mathscr{r} &= \sqrt{\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}} \\
&= \sqrt{(\vec{r} - \vec{r}\,') \cdot (\vec{r} - \vec{r}\,')} \\
&= \sqrt{(r^2 + r'^2 - 2\vec{r}\,' \cdot \vec{r})} \\
&= \sqrt{r^2 \left(1 + \frac{r'^2}{r^2} - \frac{2\vec{r}\,' \cdot \vec{r}}{r^2}\right)} \\
&= r \left(1 + \frac{r'^2}{r^2} - \frac{2\vec{r}\,' \cdot \vec{r}}{r^2}\right)^{1/2}
\end{aligned}$$

Thus

$$dV = \frac{1}{4\pi\epsilon_0} \frac{\rho(\vec{r}\,')\,d\tau'}{r\left(1 + r'^2/r^2 - 2\,\frac{\vec{r}\,'\cdot\vec{r}}{r^2}\right)^{1/2}}$$

Then an exact expression for the potential is:

$$V = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_{\text{all distribution}} \frac{\rho(\vec{r}\,')\,d\tau'}{\left(1 + r'^2/r^2 - 2\,\frac{\vec{r}\,'\cdot\vec{r}}{r^2}\right)^{1/2}}$$

Note the two terms in the denominator:

1) $\frac{r'^2}{r^2}$ magnitude $\sim \left(\frac{r'}{r}\right)^2$

2) $\frac{\vec{r}\,'\cdot\vec{r}}{r^2} = \frac{\vec{r}\,'}{r}\cdot\hat{r}$ magnitude $\sim \left(\frac{r'}{r}\right)$ ← dominant

Now a Taylor series approximation is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots\ldots$$

We can apply this with:

$n = -1/2$

$x = \left(\frac{r'}{r}\right)^2 - 2\left(\frac{\vec{r}\,'\cdot\vec{r}}{r^2}\right)$

Then:

$$\left[1+\left(\frac{r'}{r}\right)^2-2\,\frac{\vec{r}'\cdot\vec{r}}{r^2}\right]^{1/2} = 1-\frac{1}{2}\left[\left(\frac{r'}{r}\right)^2-2\,\frac{\vec{r}'\cdot\vec{r}}{r^2}\right]+\frac{3}{8}\left[\left(\frac{r'}{r}\right)^2-2\,\frac{\vec{r}'\cdot\vec{r}}{r^2}\right]^2$$

$$= 1-\underset{\hookrightarrow O(2)}{\frac{1}{2}\left(\frac{r'}{r}\right)^2}+\underset{\hookrightarrow O(1)}{\frac{\vec{r}'\cdot\vec{r}}{r^2}}+\frac{3}{8}\left[\underset{\hookrightarrow O(4)}{\left(\frac{r'}{r}\right)^4}-\underset{\hookrightarrow O(3)}{4\,\frac{\vec{r}'\cdot\vec{r}}{r^2}\left(\frac{r'}{r}\right)^2}+\underset{\hookrightarrow O(2)}{4\,\frac{(\vec{r}'\cdot\vec{r})^2}{r^4}}\right]$$

$$= 1+\underset{\text{magnitude } \frac{r'}{r}}{\frac{\vec{r}'\cdot\vec{r}}{r^2}}+\frac{1}{2}\underbrace{\left[3\,\frac{(\vec{r}'\cdot\vec{r})^2}{r^4}-\left(\frac{r'}{r}\right)^2\right]}_{\text{magnitude } \left(\frac{r'}{r}\right)^2}$$

Thus:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\int \rho(\vec{r}')\left\{1+\frac{\vec{r}'\cdot\vec{r}}{r^2}+\frac{1}{2}\left[3\,\frac{(\vec{r}'\cdot\vec{r})^2}{r^4}-\left(\frac{r'}{r}\right)^2\right]\right\}d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\int \rho(\vec{r}')\,d\tau' \quad \Big\}\ \text{monopole term}$$

$$+\ \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\int \rho(\vec{r}')\,\frac{\vec{r}'\cdot\vec{r}}{r^2}\,d\tau' \quad \Big\}\ \text{dipole term}$$

$$+\ \frac{1}{4\pi\epsilon_0}\,\frac{1}{r}\int \rho(\vec{r}')\,\frac{1}{2}\left[3\,\frac{(\vec{r}'\cdot\vec{r})^2}{r^4}-\left(\frac{r'}{r}\right)^2\right]d\tau' \quad \Big\}\ \text{quadrupole term}$$

$$+ \cdots$$

Then with

$$\hat{r} = \frac{\vec{r}}{r}$$

we get

$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}')\, d\tau' \quad \leftarrow \text{monopole} \\ &\qquad \text{charge distrib} \\ &+ \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int \rho(\vec{r}')\, \vec{r}'\cdot\hat{r}\, d\tau' \leftarrow \text{dipole} \\ &+ \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int \rho(\vec{r}')\, \frac{1}{2}\left[3\,(\vec{r}'\cdot\hat{r}) - r'^2\right] d\tau' \leftarrow \text{quadrupole} \\ &+ \cdots \end{aligned}}$$

This is the multipole expansion