Lecture 22

Thurs : HW

Fri: —

Mon: Read 3.4.1, 3.4.2

Laplace's equation

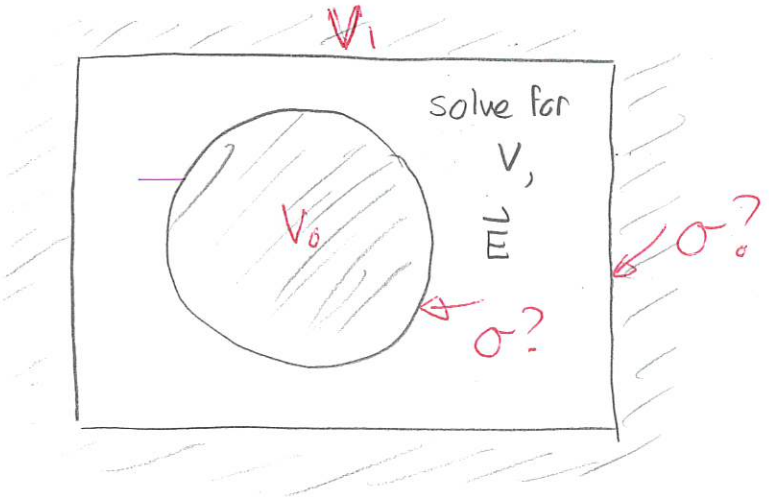
In regions where there is zero charge the electrostatic potential satisfies Laplace's equation

$$\nabla^2 V = 0$$

We can solve this subject to boundary conditions that we provide. Given a solution, we can obtain an electric field

$$\vec{E} = -\vec{\nabla} V$$

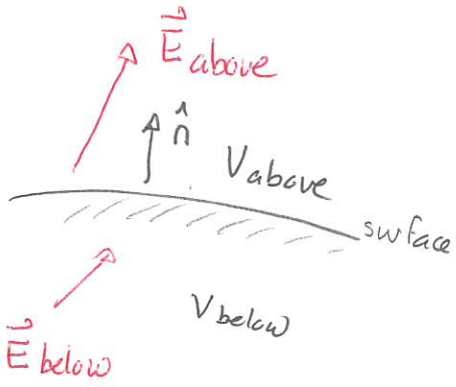
What still remains is to determine charge densities that may exist and which generate fields and potentials.



Boundary charge densities

We can obtain charge densities on a boundary provided that we know

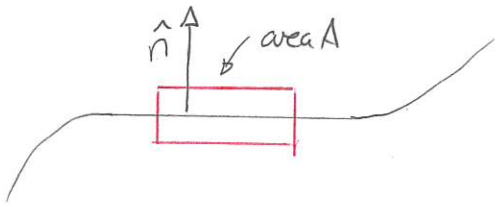
* the potential on either side OR
* the electric field on either side



To set this up, let $\hat{n}$ be a normal vector to the surface. This defines a region above the surface and a region below the surface. Let σ be the charge density on the surface. Then we apply Gauss' Law to an infinitesimal pill box. In general

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

$$= \frac{\sigma A}{\epsilon_0}$$



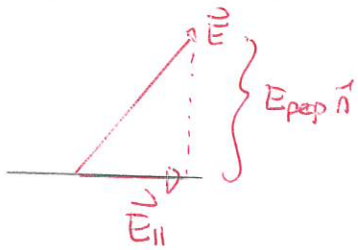
Now the sides of the pillbox can be constructed to have a vanishingly small area. Thus they do not contribute to the surface integral. So

$$\oint \vec{E} \cdot d\vec{a} = \vec{E}_{above} \cdot \hat{n} A - \vec{E}_{below} \cdot \hat{n} A = \sigma A / \epsilon_0$$

$$\Rightarrow \boxed{\vec{E}_{above} \cdot \hat{n} - \vec{E}_{below} \cdot \hat{n} = \sigma / \epsilon_0}$$

This means that the components of $\vec{E}$ perpendicular to the surface are discontinuous.

Note that in general any electric field at the surface can be expressed as:



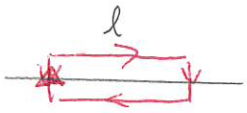
$$\vec{E} = E_{perp}\,\hat{n} + \vec{E}_{\parallel}$$

where $\vec{E}_{\parallel}$ is a component parallel to the surface. Thus

$$E_{perp\ above} - E_{perp\ below} = \sigma/\epsilon_0$$

Separately we can address the parallel components by using $\vec{\nabla}\times\vec{E} = 0$ and Stokes Law for the illustrated loop.

Then $\vec{\nabla}\times\vec{E} = 0$ implies



$$\oint_{loop} \vec{E}\cdot d\vec{l} = 0$$

$$\Rightarrow \int_{left} \vec{E}\cdot d\vec{l} + \int_{below} \vec{E}\cdot d\vec{l} + \int_{right} \vec{E}\cdot d\vec{l} + \int_{top} \vec{E}\cdot d\vec{l} = 0$$

left: vanishing by small

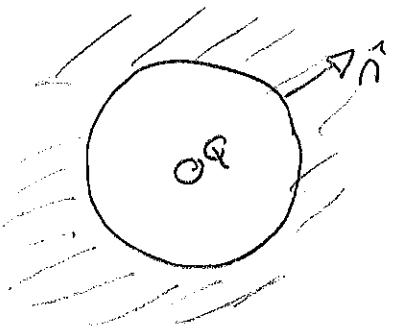
$$= 0 \qquad E_{above\parallel}\, l - E_{below\parallel}\, l = 0 \quad \Rightarrow \quad E_{above\parallel} = E_{below\parallel}$$

So the parallel components are continuous.

$$\vec{E}_{\parallel\ above} = \vec{E}_{\parallel\ below}$$

Example We can find the surface charge density in the spherical cavity situation. Inside the cavity

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$



outside $\vec{E} = 0$. letting $\hat{n} = \hat{r}$

we get

$$\vec{E}_{outside}^{\,0} \cdot \hat{n} - \vec{E}_{inside} \cdot \hat{n} = \sigma/\epsilon_0$$

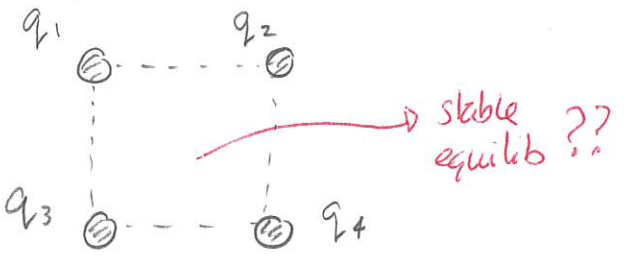
$$\Rightarrow \quad -\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} = \sigma/\epsilon_0 \quad \Rightarrow \quad \boxed{\sigma = -\frac{Q}{4\pi R^2}}$$

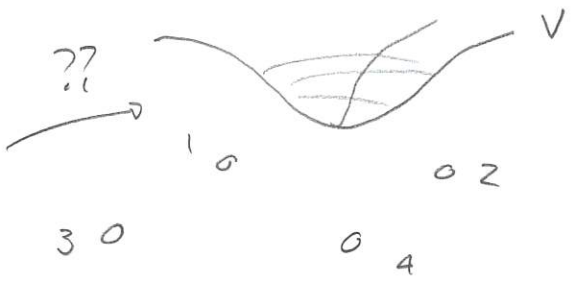
Local maxima and minima for potentials

Consider a region that is free of charge, i.e. $\rho = 0$. Then we know that inside this region

$$\nabla^2 V = 0$$

Is it possible that the potentials on the boundary can be set up so that there is a local maximum or local minimum? This would then provide a stable equilibrium point. So with four point charges:



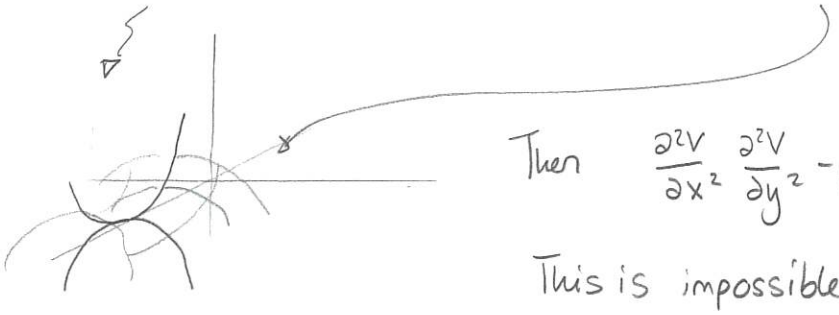


Consider a two dimensional variant of this. A local minimum requires

1) $\frac{\partial V}{\partial x} = 0$ and $\frac{\partial V}{\partial y} = 0$

2) $\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 > 0$ and $\frac{\partial^2 V}{\partial x^2} > 0$

But if $\frac{\partial^2 V}{\partial x^2} > 0$ then $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 V}{\partial y^2} < 0$



Then $\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 < 0$

This is impossible.

This provides a two dimensional example of a more general result:

> For any region in which Laplace's equation is satisfied, i.e. $\nabla^2 V = 0$, the electrostatic potential cannot have a local maximum or local minimum. These can only be attained on the boundaries of the region.

Demo: * Show Ion Traps
↳ U Oxford - how to trap ions

* Paul Trap video

Dipoles + quadrupoles

So far, exact methods for calculating electric fields or electrostatic potentials are:

1) Coulomb's Law → fields
2) Gauss' Law → fields
3) Direct calculation of electrostatic potential → potential
4) Solving Laplace's or Poisson's equations → potential

These techniques will not always succeed analytically and we need approximation techniques in many situations. One such approach in electrostatics is the multipole expansion which results in

$$V = V_{\text{monopole}} + V_{\text{dipole}} + V_{\text{quadrupole}} + \dots$$

typically diminishing size

We now present a technique for doing this in general. The technique will give:

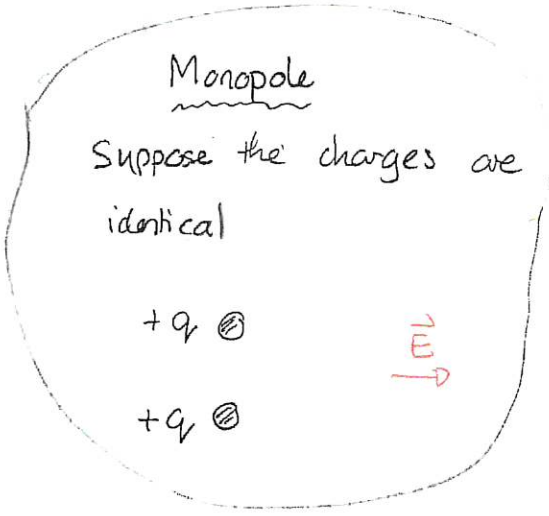
1) approximation techniques...
2) introduction to dipoles which for models for material systems

Monopoles and dipoles

Consider arrangements of two point charges and the fields that these produce at large distances from the source charges

Monopole

Suppose the charges are identical



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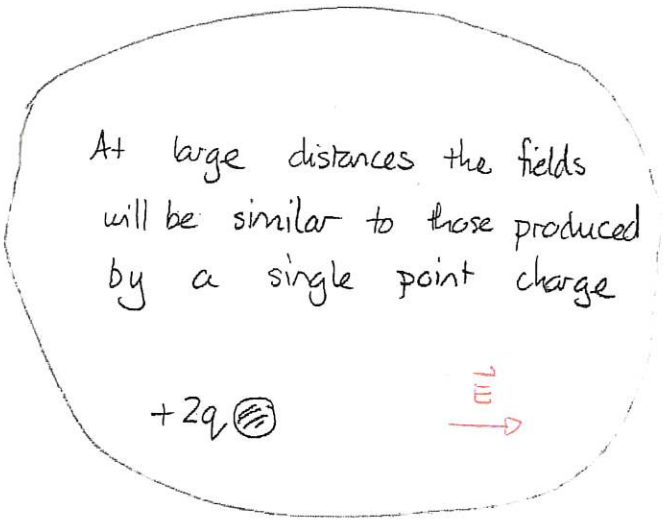
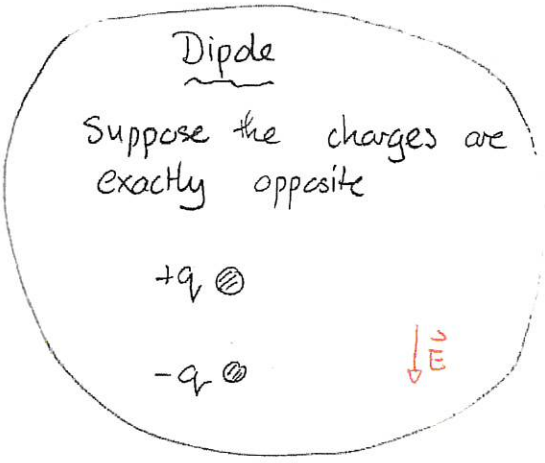
At large distances the fields will be similar to those produced by a single point charge



The two charges can then be approximated by a single point charge, or a monopole, at large distances. The approximation breaks down close to the two charges.

Dipole

Suppose the charges are exactly opposite



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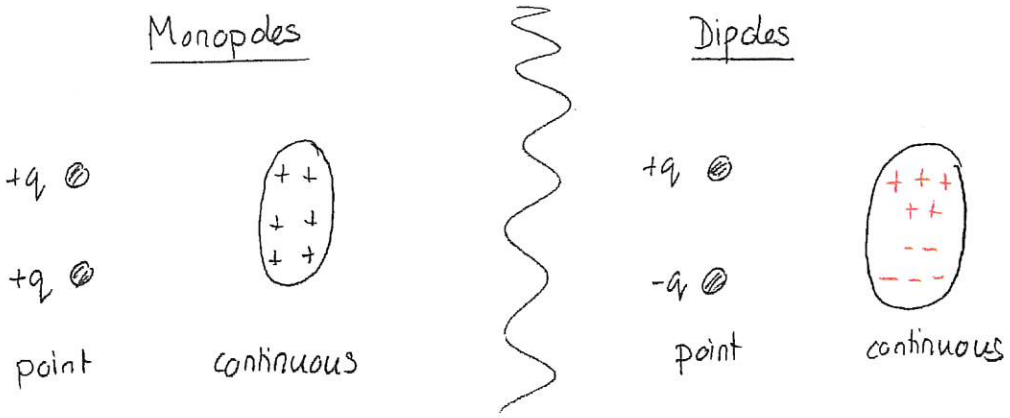
At large distances the overall charge may appear to be zero. But the net field is still not zero. This results from the two distinct charges with opposite signs. This forms a dipole and we need approximate methods for determining this dipole field

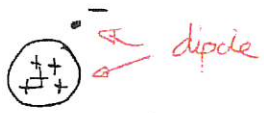
In the first case the monopole field dominates (it will be larger than the dipole field). In the second case the monopole field will be zero and the dipole field dominates

We will eventually consider continuous charge distributions that display monopole and dipole characteristics.

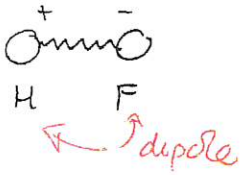


Idealized dipole distributions form models for many real charge distributions, particularly in atomic and molecular physics.

1) Potassium atom



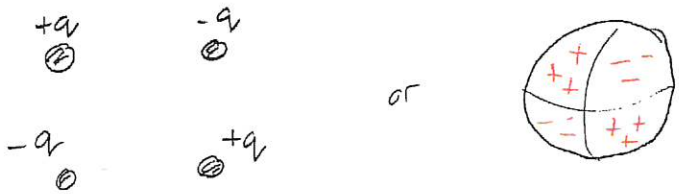
2) Polar molecules



The interactions between such atoms and molecules can be well modeled by considering these as dipoles.

Quadrupoles

A quadrupole consists of a charge arrangement of the type.



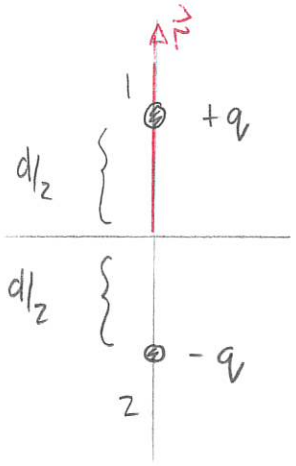
These are important in certain ion traps.

Demo * Innstruck Ion Trap
* Faistad quadrupole
* teaching topics / emag / upperclass → dipole and quadrupole

1 Point charge dipole

Consider the point charge dipole with charges located along the z-axis at $z = +d/2$ and $z = -d/2$. Determine an expression for the electrostatic potential at any point along the z axis. Approximate this when $z \gg d$

Answer:



$\vec{r} = z\hat{z}$

$\vec{r}_1' = d/2\,\hat{z}$

$\vec{r}_2' = -d/2\,\hat{z}$

$\vec{\rho}_1 = \vec{r} - \vec{r}_1'$

$= (z - d/2)\,\hat{z}$

$\vec{\rho}_2 = \vec{r} - \vec{r}_2'$

$= (z + d/2)\,\hat{z}$

$V = V_1 + V_2$

$V_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{\rho_1} = \frac{1}{4\pi\epsilon_0}\frac{q}{(z - d/2)}$

$V_2 = \frac{1}{4\pi\epsilon_0}\frac{q}{\rho_2} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z + d/2}$

$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{z - d/2} - \frac{1}{z + d/2}\right]$

$= \frac{q}{4\pi\epsilon_0}\frac{1}{z}\left\{\frac{1}{1 - \frac{d}{2z}} - \frac{1}{1 + \frac{d}{2z}}\right\}$

Then $\frac{1}{1+x} \approx 1 - x \ldots$ gives

$V \approx \frac{q}{4\pi\epsilon_0 z}\left[\frac{d}{2z} + \frac{d}{2z}\right] = \frac{qd}{4\pi\epsilon_0 z^2}$ gives as $\frac{1}{z^2}$ not $\frac{1}{z}$