Weds: 3.4.1 → 3.4.3

Thurs: HW 13

Laplace's equation

Recall that in electrostatics a source charge density $\rho(\vec{r})$ produces

* electric field $\vec{E}$ s.t. $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$
* electrostatic potential V s.t. $\vec{E} = -\vec{\nabla}V$

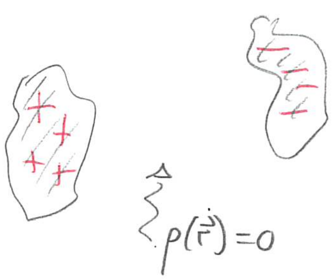
Combining these two equations relates the electrostatic potential to the source charges

$$-\vec{\nabla}\cdot\vec{\nabla}V = \rho/\epsilon_0 \Rightarrow \boxed{\nabla^2 V = -\frac{\rho(\vec{r})}{\epsilon_0}}$$

This is Poisson's equation. This gives the scheme:

Known charge density $\rho(\vec{r})$ → Solve Poisson's eqn $\nabla^2 V = -\rho/\epsilon_0$ → Determine electric field $\vec{E} = -\vec{\nabla}V$

A very important special case considers a region with no charge. Here $\rho(\vec{r}) = 0$ and then:



$$\boxed{\begin{array}{l}\text{In a region with no charge, } \rho(\vec{r})=0 \\ \text{and} \quad \nabla^2 V = 0\end{array}}$$

This is Laplace's equation

Laplace's equation between parallel plates

An infinite plane conducting plate at $x = 0$ is held at potential V_0. Another at $x = L$ is held at potential V_1. There is no charge between them.

a) Provide a general solution to Laplace's equation between the plates.

b) By matching the solutions at the plates, determine a unique expression for the potential between the plates.

c) Use the potential to find the electric field between the plates.

Answer: a)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

V can only depend on x. So Laplace's equation becomes:

$$\frac{d^2 V}{dx^2} = 0$$

and the solution to this is

$$V = \alpha x + \beta.$$

b) At $x = 0$ $\quad V = \beta = V_0 \Rightarrow \beta = V_0$

At $x = L$ $\quad V = \alpha L + \beta = V_1 \Rightarrow \alpha L = V_1 - \beta$

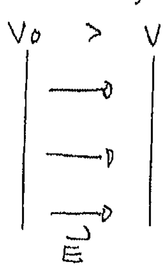
$$= V_1 - V_0$$

$$\Rightarrow \alpha = \frac{V_1 - V_0}{L}$$

So

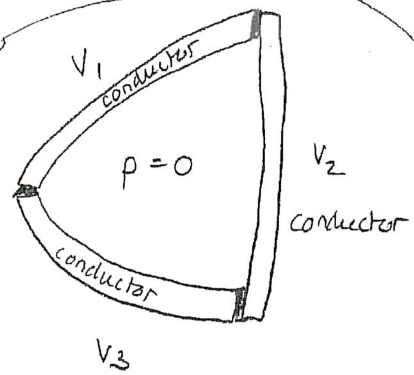
$$V(x) = \frac{V_1 - V_0}{L} x + V_0$$

c) $$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{x} - \cancel{\frac{\partial V}{\partial y}}\hat{y} - \cancel{\frac{\partial V}{\partial z}}\hat{z} = \frac{V_1 - V_0}{L}\hat{x}$$



$$\Rightarrow \vec{E} = \frac{V_0 - V_1}{L}\hat{x}$$

This illustrates a general strategy for finding fields using Laplace's equation



A charge free region is surrounded by surfaces at known fixed potentials

→

Solve Laplace's equation

$$\nabla^2 V = 0$$

in the region. This gives a non-unique general solution

↓

Apply boundary conditions = set general solution equal to potential at boundaries. This gives a unique potential $V(\vec{r})$ that matches the boundary conditions

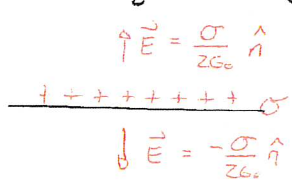
↙

Determine field

$$\vec{E} = -\vec{\nabla} V$$

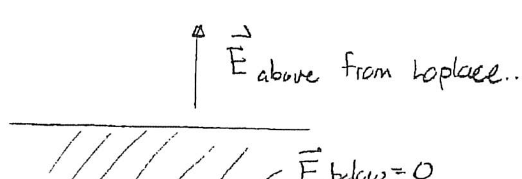
Charge densities

How can we determine the charge densities on the conducting surfaces? In general if we can obtain the field on either side then we can find the surface charge density. For an infinite sheet we can see:



$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

This is true in general. For a conductor $\vec{E}_{below} = 0$



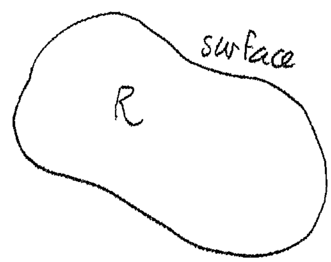
$$\frac{\sigma}{\epsilon_0}\hat{n} = \vec{E}_{above} \quad \text{gives } \sigma$$

Uniqueness of the solution to Laplace's equation

Consider a charge free region on whose boundary the potential is specified. Is there a unique solution to Laplace's equation in this region that matches on the boundary? A general result is:

<u>Theorem</u>: Suppose that $\rho = 0$ within a closed surface and the potential is specified everywhere on the boundary of the surface. Then there is at most one solution to Laplace's equation in the region that matches the conditions on the boundary.

<u>Proof</u>: Consider a region R with bounding surface S. Suppose that there are two solutions to Laplace's equation within the region. Denote these $V_1(\vec{r})$, $V_2(\vec{r})$. Let



$$\psi(\vec{r}) = V_2(\vec{r}) - V_1(\vec{r})$$

be the difference. Then:

1) $\psi(\vec{r}) = 0$ on the surface S since both $V_1(\vec{r})$, $V_2(\vec{r})$ match on S

2) Within the region $\nabla^2 \psi(\vec{r}) = \nabla^2 V_2(\vec{r}) - \nabla^2 V_1(\vec{r}) = 0$.

Thus

$$\int_R \psi \, \nabla^2 \psi \, d\tau = 0 \Rightarrow \int_R \psi \, \vec{\nabla} \cdot \vec{\nabla} \psi \, d\tau = 0$$

Then integration by parts gives:

$$\int_R \psi \, \vec{\nabla} \cdot \vec{\nabla} \psi \, d\tau = -\int_R \vec{\nabla} \psi \cdot \vec{\nabla} \psi \, d\tau + \oint_S \psi \, \vec{\nabla} \psi \cdot d\vec{a}$$

$$\Rightarrow \quad 0 = -\int_R |\vec{\nabla}\psi|^2 \, d\tau + \int 0 \, \vec{\nabla}\psi \cdot d\vec{a}$$

$\hookrightarrow \psi = 0$ on surface.

$$\Rightarrow \quad 0 = -\int |\vec{\nabla}\psi|^2 d\tau.$$

Then the integrand must be zero. Thus $\vec{\nabla}\psi = 0$ everywhere inside the region. So $\psi = \text{constant}$ inside R. Now ψ is continuous inside R and $\psi = 0$ on the boundary $\Rightarrow \psi = 0$ everywhere inside R. So

$V_2(\vec{r}) = V_1(\vec{r})$ inside R. ∎

This then permits a variety of strategies for finding potentials. We could attempt a brute force solution of the partial differential equation and then match it on the boundaries.

Alternatively any other strategy that gives V in the region that

1) satisfies $\nabla^2 V = 0$

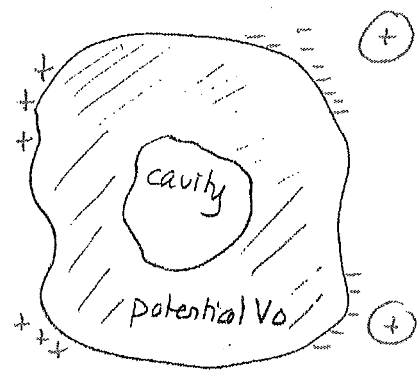
2) satisfies V on the boundaries

is a solution and since solutions are unique, this is the unique solution

Cavities inside conductors

We can apply the uniqueness of the solutions to Laplace's equation to determine electrostatic potential and electric fields within a cavity inside a conductor. Remember that:

1) the potential inside the conductor is the same throughout the conductor. This means that the potential is the same everywhere on the cavity boundary.



2) the charge density in the cavity is zero. Thus the potential inside the cavity satisfies $\nabla^2 V = 0$

Then one possible solution to Laplace's equation inside the cavity is

$$V = V_0 \equiv \text{constant}$$

We can check that this is correct as follows:

1) $\nabla^2 V = \nabla^2 V_0 = 0$ since $V_0 = $ const. ✓
2) on the boundary $V = V_0$ ✓

Then this is a solution to Laplace's equation that matches on the boundary. But such a solution is unique. Thus we have found the unique potential inside the cavity:

$$V = V_0 \equiv \text{constant}$$

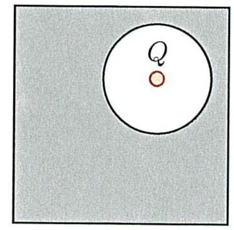
The electric field is given by $\vec{E} = -\vec{\nabla} V = -\vec{\nabla} V_0 = 0$. Thus

$$\vec{E} = 0$$

inside the cavity.

2 Charge in a cavity

A conductor contains a spherical cavity with radius R. A point particle with charge Q is placed at the center of the cavity.



a) Suppose that the electrostatic potential on the conductor is V_0. Determine the electrostatic potential and electric field everywhere inside the cavity.

b) Use Gauss' law to determine the total charge on the surface of the cavity.

Answer: a) Inside the cavity

$$\nabla^2 V = -\rho/\epsilon_0$$

$\rho \rightarrow$ point charge @ Q

means that the potential satisfies the same equation as for a point charge. On the boundary $V = V_0 = \text{const}$. Thus a possible solution is

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + C$$

where $r =$ distance from pt charge

$C =$ constant.

The term $\frac{1}{4\pi\epsilon_0} Q/r$ gives $\nabla^2\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{r}\right) = -\rho/\epsilon_0$.

We need to match on the boundary.

$$\left.\begin{matrix} r = R \\ V = V_0 \end{matrix}\right\} \Rightarrow \quad V_0 = \frac{1}{4\pi\epsilon_0}\frac{Q}{R} + C \quad \Rightarrow \quad C = V_0 - \frac{1}{4\pi\epsilon_0} Q/R$$

Thus

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} - \frac{1}{4\pi\epsilon_0}\frac{Q}{R} + V_0$$

is the unique solution

This follows because

1) V satisfies Poisson's eqn inside the cavity

2) V matches on the boundary.

The uniqueness of the solution implies V = exact/unique soln.

The field is

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

inside cavity.

b) Consider a Gaussian surface just inside the conductor
Then $\vec{E} = 0$

$$\Rightarrow \oint \vec{E} \cdot d\vec{a} = 0$$

$$\Rightarrow \frac{q_{enc}}{\epsilon_0} = 0$$

$$\Rightarrow q_{enc} = 0 \Rightarrow Q + Q_{surface} = 0 \Rightarrow Q_{surface} = -Q$$