Phys 311 2020

Lecture 20

Fri. Exam I

* Covers all material so far up to including HW 12

* ~~Study 2018 [3] Exam I 2019 [3]~~ → was a 1:15 class

* Study 2019 Ex I 50 min class

  2017 Ex I 1:15 min class

Bring: Half of letter sheet single side

Given: Copies of front/back text – formulas

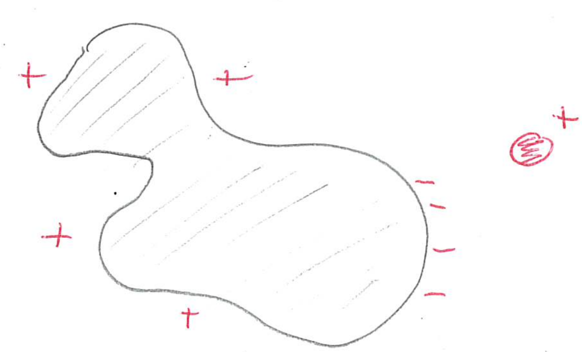
Conductors

For a perfect conductor:

1) field everywhere inside: $\vec{E} = 0$

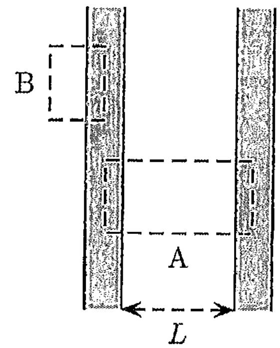
2) excess charge only resides on the surface

3) entire conductor is at the same potential i.e. equipotential



1 Parallel plate capacitor

A parallel plate capacitor consists of two conducting plane sheets that are parallel and separated by distance L. Assume that the plates extend infinitely. Excess charge is added to each plate; an equal amount is added to each plate but the charges are opposite.



a) Excess charge is added to each plate. Indicate the surfaces on which excess could reside.

b) Provide a symmetry argument that constrains the direction of the electric field between the plates and beyond them.

c) Use a Gaussian pillbox of the form illustrated by A to relate the surface charge densities on the two inner surfaces.

d) Assuming the the charge density on the left plate is equal in magnitude but opposite in sign to that on the right plate, relate the charge densities on the outer surfaces.

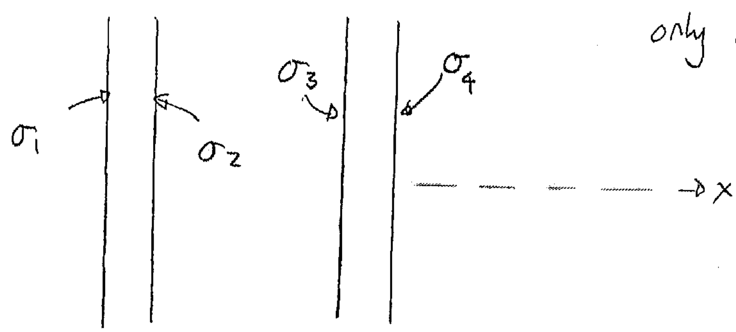
e) Use the fields produced by infinite charged sheets within each conductor to determine the charge densities on the outer surfaces.

f) Use a Gaussian pillbox of the form illustrated by B to determine the electric field beyond the outer surfaces.

g) Determine the electric field between the plates.

h) Determine the electrostatic potential difference between the plates.

Answer: a)



only on surfaces

b) We can rotate through any angle about the indicated x axis and the charge distribution remains same. So $\vec{E}$ cannot have components parallel to plates.

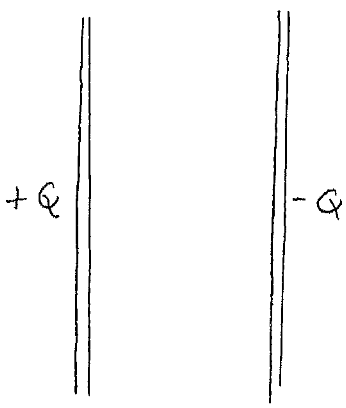
$$\vec{E} = E_x(x)\,\hat{x}$$

Capacitors

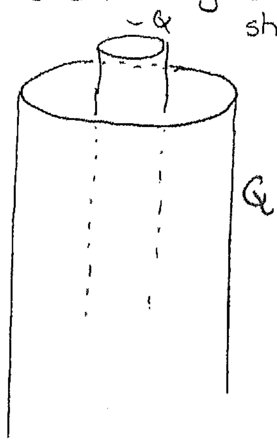
A capacitor consists of two conductors onto which charge can be placed. In principle the conductors could have arbitrary shapes. We will mostly consider more symmetrical arrangements.

In general a capacitor will be forced to carry equal and opposite charge on each conductor

parallel plate



concentric cylindrical structures



We would like to relate the charge Q to the potential difference across the two conductors. For example for the solid sphere /shell arrangement

$$Q = \underbrace{4\pi\epsilon_0 \left(\frac{1}{a} - \frac{1}{b}\right)^{-1}}_{\text{only depends on the configuration of the shells ..}} \Delta V$$

c) The box has 6 sides

front / back $\quad \vec{E}\cdot d\vec{a} = 0 \quad$ since $\vec{E}$ is $\rightarrow$ or $\leftarrow$ $\quad d\vec{a}$ in/out

top / bottom $\quad \vec{E}\cdot d\vec{a} = 0 \quad$ since $\vec{E}$ is $\rightarrow$ or $\leftarrow$ $\quad d\vec{a}$ is $\uparrow$ or $\downarrow$

left / right $\quad \underset{\text{conductor}}{\vec{E} = 0} \Rightarrow \vec{E}\cdot d\vec{a} = 0.$

vertical area box

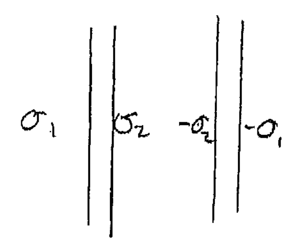
$$\text{So} \quad \oint \vec{E}\cdot d\vec{a} = 0 \Rightarrow q_{enc} = 0 \Rightarrow \sigma_2 A + \sigma_3 A = 0$$

$$\Rightarrow \sigma_2 = -\sigma_3$$

d) We have $\quad \sigma_1 + \sigma_2 = -(\sigma_3 + \sigma_4)$

$$\Rightarrow \sigma_1 + \sigma_2 = -\sigma_3 - \sigma_4$$

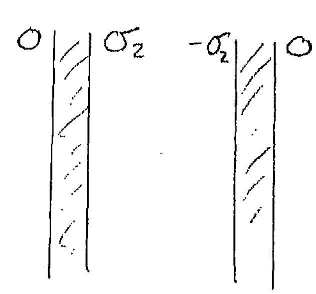
$$\Rightarrow \sigma_4 = -\sigma_1 + \cancel{\sigma_2 + \sigma_3} \quad \Rightarrow \sigma_4 = -\sigma_1$$



e) Consider left conductor

$$\vec{E} = \frac{\sigma_1}{2\epsilon_0}\hat{x} - \cancel{\frac{\sigma_2}{2\epsilon_0}\hat{x}} - \left(\cancel{\frac{-\sigma_2}{2\epsilon_0}\hat{x}}\right) - \left(\frac{-\sigma_1}{2\epsilon_0}\hat{x}\right) = 0$$

$$\Rightarrow \frac{\sigma_1}{\epsilon_0}\hat{x} = 0 \quad \Rightarrow \sigma_1 = 0$$



f) Only the left side contributes to $\oint \vec{E}\cdot d\vec{a}$. If the area is A_B. Then this gives

$$\oint \vec{E}\cdot d\vec{a} = -E_x(x)\,A_B = \frac{q_{enc}}{\epsilon_0}$$

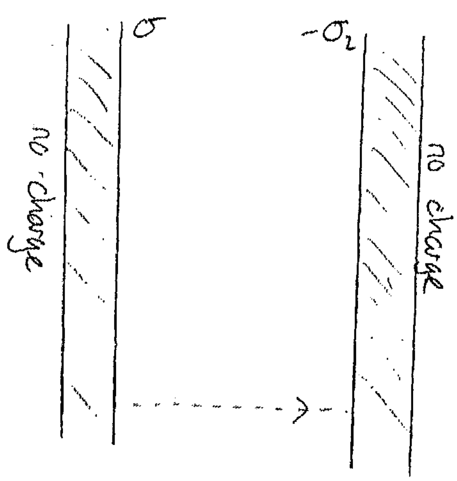
But $q_{enc}=0 \Rightarrow E_x = 0$ outside the plates.

g) The two inner surfaces each contribute

$$\frac{\sigma_2}{2\epsilon_0}\hat{x}$$

$$\Rightarrow \vec{E} = 2\,\frac{\sigma_2}{2\epsilon_0}\hat{x} \Rightarrow \vec{E} = \frac{\sigma_2}{\epsilon_0}\hat{x}$$

Rewrite the surface charge density as $\sigma \equiv \sigma_2$



$$\vec{E} = \begin{cases} 0 & \text{outside} \\ \frac{\sigma}{\epsilon_0}\hat{x} & \text{between plates} \end{cases}$$

h) Follow the indicated path $\Delta V = -\int \vec{E}\cdot d\vec{l}$ $\quad d\vec{l} = dx\,\hat{x}$

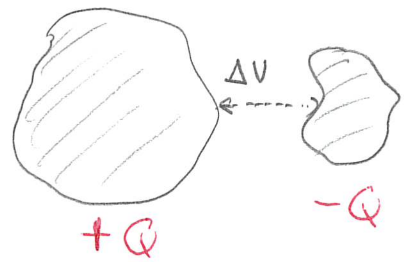
$$= -\frac{\sigma}{\epsilon_0}\int_0^L dx = -\frac{\sigma L}{\epsilon_0}$$

$$\Rightarrow V_{left} - V_{right} = -\frac{\sigma L}{\epsilon_0}$$

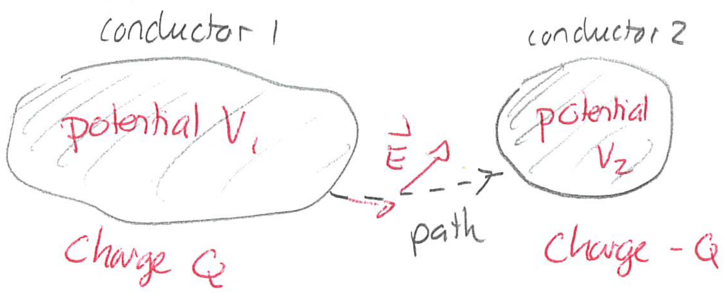
Capacitance

This example and the previous example of concentric conductors illustrate an important fact about capacitors.

> If a capacitor consists of two conductors with equal and opposite charges on each conductor then the electrostatic potential difference between the conductors is proportional to the magnitude of the charge on each
>
> So
>
> $$\Delta V = \text{const} \times Q$$



This can be demonstrated via a general argument involving two conductors with an "equal but opposite" charge configuration.



1) the potential is the same everywhere on any single conductor. So $\Delta V = V_2 - V_1$ only depends on the potential on each surface

2) the potential is

$$\Delta V = -\int_{\text{path}} \vec{E} \cdot d\vec{l}$$

where $\vec{E}$ is the field produced by the charges. This depends on Q and the configuration.

3) $\vec{E}$ is proportional to Q

$\Rightarrow$ ΔV is proportional to Q.

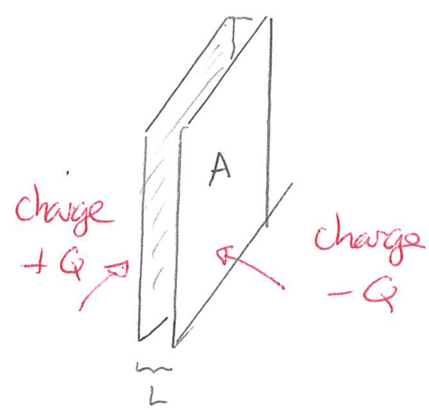
Thus we arrive at:

For a pair of conductors that carry equal and opposite charges (with magnitude Q) $Q = C\Delta V$ where ΔV is the potential difference between the conductors and C is the capacitance.

The units of capacitance are Farads $F = C/V$

Example: A parallel plate capacitor consists of two sheets that are very closely spaced. Suppose the separation is L and the area A

If the separation is small compared to the area dimensions then, between the plates the field is that of an infinite sheet. So



$$\Delta V = \frac{\sigma}{\epsilon_0} L$$

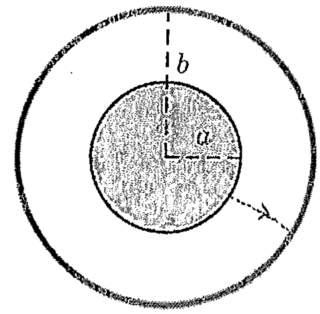
But $\sigma = Q/A$

$$\Rightarrow \quad \Delta V = \frac{QL}{\epsilon_0 A} \quad \Rightarrow \quad Q = \frac{\epsilon_0 A}{L}\Delta V \quad = C\Delta V$$

where $C = \dfrac{\epsilon_0 A}{L}$

1 Capacitance

A solid spherical conductor is situated within a hollow spherical shell. The sphere carries charge Q and the shell carries charge $-Q$.



a) Determine the capacitance of the arrangement.

b) Determine the energy stored in the electric field using

$$W_{\text{ext}} = \frac{\epsilon_0}{2}\int \mathbf{E}\cdot\mathbf{E}d\tau.$$

c) Show that the energy stored in the capacitor is $\frac{1}{2}C(\Delta V)^2$.

Answer: a) The usual Gauss' Law argument gives between the conductors

$$4\pi r^2 E_r(r) = \frac{q_{enc}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

$$\Rightarrow \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \qquad a \leq r \leq b$$

Now consider the illustrated path from $r=a$ to $r=b$. Along this path $d\vec{l} = dr\,\hat{r}$ $\quad a \leq r \leq b$, and

$$\Delta V = -\int \vec{E}\cdot d\vec{l}$$

$$\Rightarrow \quad V(b)-V(a) = -\int_a^b \frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

$$\Rightarrow \quad V(b)-V(a) = \frac{Q}{4\pi\epsilon_0}\left.\frac{1}{r}\right|_a^b = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{b}-\frac{1}{a}\right]$$

Thus the magnitude of ΔV is

$$|\Delta V| = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right] \Rightarrow \quad Q = \frac{4\pi\epsilon_0}{[1/a - 1/b]}|\Delta V|$$

$$\Rightarrow \quad C = \frac{4\pi\epsilon_0}{1/a - 1/b}$$

b) $$W_{ext} = \frac{\epsilon_0}{2} \int_a^b dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}$$

$$= \frac{\epsilon_0}{2} \frac{Q^2}{16\pi^2\epsilon_0^{\cancel{2}}} \int_a^b \frac{1}{r^2} dr \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta}_{4\pi}$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

c) $$\frac{1}{2} C (\Delta V)^2 = \frac{1}{2} \frac{4\pi\epsilon_0}{(1/a - 1/b)} \left[\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)\right]^2$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

This matches part b).

One can show generally that

$$\boxed{\text{The energy stored in a capacitor is} \quad W = \frac{1}{2} C(\Delta V)^2}$$