Tues: HW

Weds: Read. 2.5.4.

Conductors

A conductor is a material, usually a metal, in which charge can move without resistance. A perfect conductor is a material which contains an unlimited supply of charged particles which are free to move without any resistance.

In electrostatics a conductor is constrained by the condition that charges cannot move. Thus there cannot be an electric field inside the conductor. If there were, then the charges within the conductor would accelerate, and would not be at rest. Thus

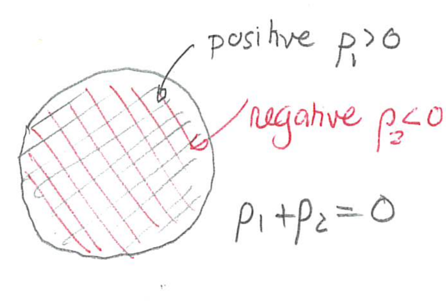
$$\boxed{\text{Inside any conductor, the electric field } \vec{E} = 0}$$

Then Gauss' Law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \Rightarrow \rho = 0$ inside. Thus

$$\boxed{\text{Inside any conductor, } \rho = 0}$$

This does not imply that there are no charges present inside the conductor. Rather the density of negatively charged particles is exactly the opposite of the density of positively charged particles. Thus

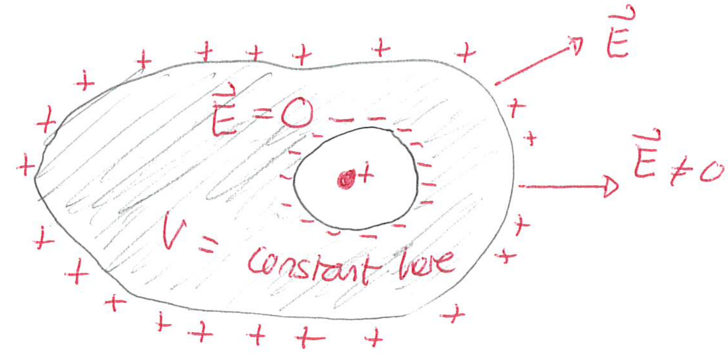
$$\boxed{\text{Any excess charge on a conductor must reside on its surface.}}$$



Also $\quad \Delta V = -\int \vec{E} \cdot d\vec{l} = 0$ between any two points in the conductor. Thus

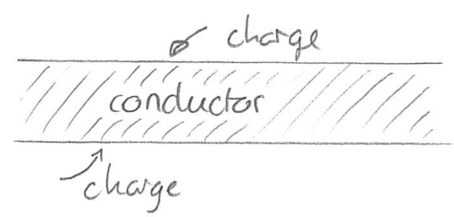
$$\boxed{\text{Inside a conductor } V \text{ is a constant throughout. The entire conductor is an equipotential.}}$$

Schematically



Infinite Conducting Sheet

An infinite conducting sheet is a plane with non-zero thickness that contains a perfect conductor. Charge can be placed on the plane. We want to determine fields and potentials produced by this charge. This will be important for parallel plate capacitors. The basic ingredients in the analysis are:

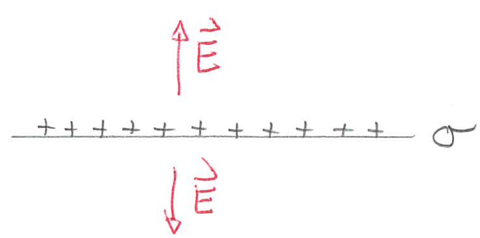


1) $\vec{E} = 0$ inside conductor

2) excess charge must reside on the surfaces

3) A single uniformly charged infinite sheet with charge density σ produces a field

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \underbrace{\hat{n}}_{\text{normal away from surface}}$$



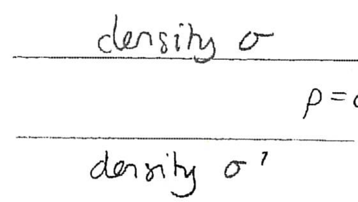
1 Infinite conducting sheet

A flat conducting sheet has non-zero thickness. It is found that one surface carries a uniform charge with density σ.

a) Determine the charge density on the other surface.

b) Use Gauss' Law to determine the electric field at any location beyond the sheet.

Answer: a) Suppose that each surface carries the indicated charge densities. Then these produce fields inside the conductor



$$\vec{E} = \vec{E}_{upper} + \vec{E}_{lower} = -\frac{\sigma}{2\epsilon_0}\hat{z} + \frac{\sigma'}{2\epsilon_0}\hat{z} = \frac{1}{2\epsilon_0}(\sigma' - \sigma)\hat{z}$$

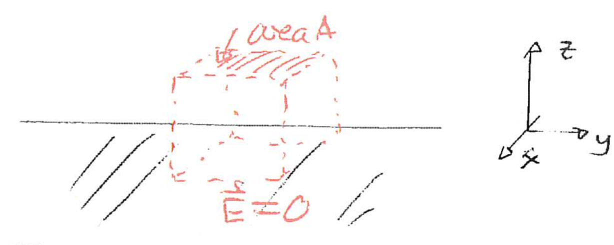
This is only zero if $\sigma = \sigma'$

Thus $\boxed{\text{The fact that } \vec{E} = 0 \text{ and } \rho = 0 \text{ inside the conductor is only possible if the charge density is same on each surface}}$

b) Use Gauss' Law. By the usual symmetry

$$\vec{E} = E_z(z)\hat{z}$$

outside the surface.



Using the usual Gaussian pillbox with sides parallel to $\vec{E}$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

$$\int \vec{E} \cdot d\vec{a} = \begin{cases} 0 & \text{on sides} \quad \text{because} \quad \vec{E} \cdot d\vec{a} = 0 \\ 0 & \text{on bottom} \quad \text{"} \quad \vec{E} = 0 \end{cases}$$

On top: $\quad d\vec{a} = dx'dy'\,\hat{z}$

$$\vec{E}\cdot d\vec{a} = E_z(z)\,dx'dy'$$

$$\int \vec{E}\cdot d\vec{a} = E_z(z)\,A = \frac{q_{enc}}{\epsilon_0}$$

But $\quad q_{enc} = \sigma A$

$$\Rightarrow \quad E_z(z) = \frac{\sigma}{\epsilon_0}$$

By symmetry we get a similar result beneath the plate. $\boxtimes$

Thus

> The electric field produced by a single conducting sheet with charge density σ on each side is
>
> $$\vec{E} = \begin{cases} 0 & \text{inside} \\ \dfrac{\sigma}{\epsilon_0}\,\hat{n} & \text{outside} \end{cases}$$

Note the consistency of results.

upper   lower

$$\uparrow \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0}\uparrow = \frac{\sigma}{\epsilon_0}\uparrow$$

upper σ + + +

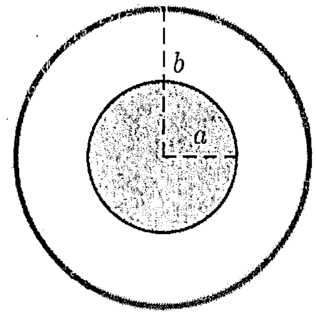
$$\downarrow \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0}\uparrow = 0$$

lower σ + + +

$$\downarrow \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0}\downarrow = \frac{\sigma}{\epsilon_0}\downarrow$$

1. Concentric conducting spheres

A solid spherical conductor is situated within a hollow conducting spherical shell. The sphere carries charge Q and the shell carries charge $-Q$.



a) Determine the electric field at all points.

b) Determine the electrostatic potential at all points such that $V = 0$ at infinity.

c) Determine the energy stored in the charge arrangement.

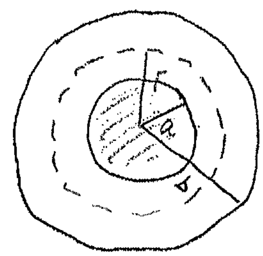
d) Determine the potential difference between the spheres. How does it depend on the charge on either object?

Answer: a) By symmetry $\vec{E} = E_r(r)\,\hat{r}$. We use a sphere of radius r as a Gaussian sphere and we use this in three situations:

i) $r < a$

ii) $a < r < b$ (illustrated)

iii) $b < r$



In all situations

$$\left.\begin{array}{c} r' = r \\ 0 \leq \theta' \leq \pi \\ 0 \leq \phi' \leq 2\pi \end{array}\right\} \qquad d\vec{a} = r'^2 \sin\theta'\, d\theta'\, d\phi'\, \hat{r} = r^2 \sin\theta'\, d\theta'\, d\phi'\, \hat{r}$$

So $\vec{E}\cdot d\vec{a} = E_r(r)\, r^2 \sin\theta'\, d\theta'\, d\phi'$

$$\oint \vec{E}\cdot d\vec{a} = \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\, E_r(r)\, r^2 \sin\theta'$$

$$= E_r(r)\, r^2 \underbrace{\int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \sin\theta'}_{4\pi} \Rightarrow \oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E_r(r)$$

Gauss' Law gives $\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$

$$\Rightarrow \quad E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2}$$

Case i). $r < a$ $\quad q_{enc} = 0 \quad \Rightarrow \quad E_r(r) = 0$

ii) $a < r < b \quad q_{enc} = Q \quad \Rightarrow \quad E_r(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$

iii) $b < r \quad q_{enc} = 0 \quad \Rightarrow \quad E_r(r) = 0$

Thus

$$\vec{E} = \begin{cases} 0 & r < a \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} & a < r < b \\ 0 & b < r \end{cases}$$

b) In general

$$\Delta V = -\int \vec{E} \cdot d\vec{l}$$

between any two points. Then outside the shell $(r > b)$ $\vec{E} = 0$

Thus

$$\Delta V = 0$$

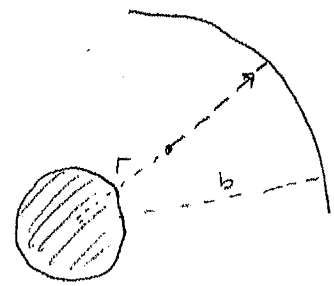
outside the shell But $V = 0$ at $r = \infty$ implies.

$$V(r) = 0 \qquad r > b$$

Thus at $r = b$, $V = 0$

Then, between the shells $\vec{E} = \frac{Q}{4\pi\epsilon_0 r}\hat{r}$ So consider a path from r to b

Here $d\vec{l} = dr'\hat{r}$



$r < r' < b$

So

$$\Delta V = V(b) - V(r) = -\int \vec{E}\cdot d\vec{l}$$

$$= -\int_r^b \frac{1}{4\pi\epsilon_0}\frac{Q}{r'^2}\,dr' = \frac{Q}{4\pi\epsilon_0 r'}\Big|_r^b$$

$$\Rightarrow \cancel{V(b)}^{\,0} - V(r) = \frac{Q}{4\pi\epsilon_0 b} - \frac{Q}{4\pi\epsilon_0 r}$$

$$\Rightarrow V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} - \frac{1}{4\pi\epsilon_0}\frac{Q}{b}. \qquad a < r < b$$

Inside the inner sphere $V = \text{constant} = V(a)$. From the above

$$V(a) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{a} - \frac{Q}{b}\right)$$

So we have

$$V(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) & r \le a \\ \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right) & a \le r \le b \\ 0 & b \le r \end{cases}$$

c) $$W_{ext} = \frac{\epsilon_0}{2}\int_{\text{all space}} \vec{E}\cdot\vec{E}\,d\tau$$

This is only non-zero between the sphere + shell. So

$$W_{ext} = \frac{\epsilon_0}{2}\int_a^b dr\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\; r^2\sin\theta \;\frac{Q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}$$

$$= \frac{\epsilon_0}{2}\frac{Q^2}{16\pi^2\epsilon_0^2}\int_a^b \frac{1}{r^2}dr\;\underbrace{\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\sin\theta}_{4\pi}$$

$$= \frac{1}{8\pi\epsilon_0}Q^2\left[-\frac{1}{r}\right]_a^b$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

d) $\Delta V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$. This is proportional to Q.