Phys 311 2020

Lecture 18

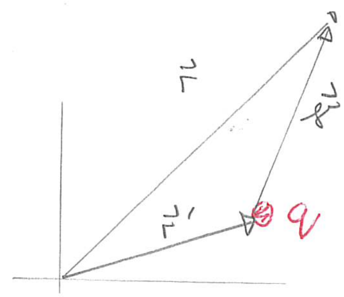
Mon: 2.5.1 – 2.5.3

Tues: HW

Direct calculation of electrostatic potential

The method for computing the electrostatic potential due to a single point charge can be extended via superposition to collections of charges. The basic ingredient is:

> The electrostatic potential at location $\vec{r}$ produced by a point charge at location $\vec{r}\,'$ is:
>
> $$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$
>
> where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and q is charge of the point charge.

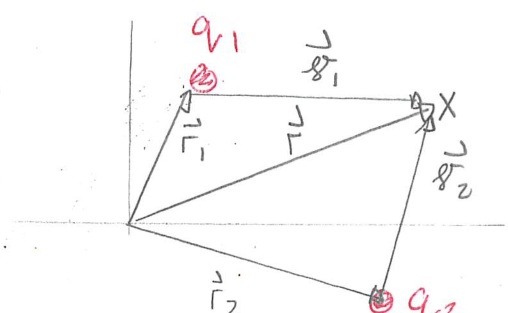


With multiple point sources we can use the superposition principle to simply add potentials produced by individual sources.

$$V(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r}) + \dots$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1}{\mathscr{r}_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{\mathscr{r}_2} + \dots$$

This is scalar addition.



We can extend this to continuous distributions by decomposing and integrating. Then for a volume charge distribution

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{\mathscr{r}} \, d\tau'}$$

For a surface charge distribution

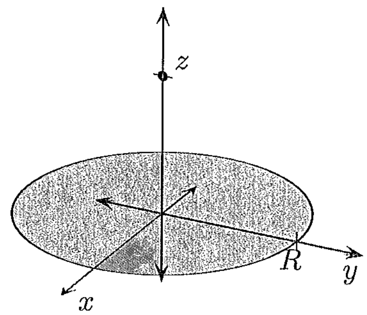
$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}\,')}{\mathscr{r}} \, da'}$$

and for a line charge distribution

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}\,')}{\mathscr{r}} \, dl'}$$

1 Potential above a disk

A disk with radius R lies in the xy plane. The surface charge density in the disk is $\sigma(\mathbf{r}') = \sigma(s')$.

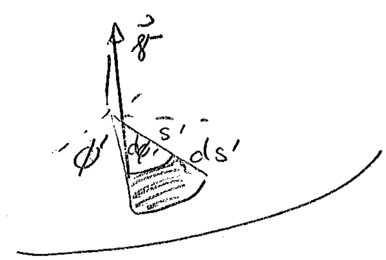


a) Determine a general expression for the electrostatic potential at any point along the z axis.

b) Evaluate the expression when the disk is uniformly charged. Rewrite this in terms of the total charge Q on the disk.

c) Check that the expression for $z \gg R$. Does this agree with what you would expect?

Answer: a) ① Break into pieces + integrate. Consider the contribution from the segment



$$\left.\begin{array}{l} s' \to s' + ds' \\ \phi' \to \phi' + d\phi' \end{array}\right\} \quad da' = s'ds'd\phi'$$

- The enclosed charge is

$$dq = \sigma(\vec{r}')\,da' \Rightarrow \quad dq = \sigma s'ds'd\phi'$$

② Then $$\left.\begin{array}{l} \vec{r} = z\hat{z} \\ \vec{r}' = s'\hat{s} \end{array}\right\} \quad \vec{\mathscr{r}} = \vec{r} - \vec{r}' = z\hat{z} - s'\hat{s}$$

Thus $$\mathscr{r} = \sqrt{\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}} = \sqrt{z^2 + s'^2}$$

Then the contribution is:

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{\mathscr{r}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma(s')}{z^2+s'^2}\, s'ds'd\phi'$$

(3) The total potential is

$$V = \frac{1}{4\pi\epsilon_0} \int_0^R ds' \int_0^{2\pi} d\phi' \; s' \frac{\sigma(s')}{\sqrt{z^2+s'^2}}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^R \frac{s'\sigma(s')}{\sqrt{z^2+s'^2}} 2\pi \quad \Rightarrow \quad V = \frac{1}{2\epsilon_0} \int_0^R \frac{s'\sigma(s')}{\sqrt{z^2+s'^2}} ds'$$

b) Here σ is constant

$$V = \frac{\sigma}{2\epsilon_0} \int_0^R \frac{s'}{\sqrt{z^2+s'^2}} ds'$$

$$= \frac{\sigma}{2\epsilon_0} \left(z^2+s'^2\right)^{1/2} \Big|_0^R$$

$$= \frac{\sigma}{2\epsilon_0} \left\{\sqrt{z^2+R^2} - z\right\}$$

Now the total charge is $Q = \sigma \pi R^2 \Rightarrow \sigma = \frac{Q}{\pi R^2}$

Thus

$$V = \frac{Q}{2\pi\epsilon_0} \frac{1}{R^2} \left\{\sqrt{z^2+R^2} - z\right\} \Rightarrow V = \frac{Q}{2\pi\epsilon_0} \frac{z}{R^2} \left\{\sqrt{1+R^2/z^2} - 1\right\}$$

c) We need a Taylor series

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \ldots$$

$$\Rightarrow \sqrt{1+R^2/z^2} - 1 \approx \cancel{1} + \frac{1}{2}\frac{R^2}{z^2} - \cancel{1} = \frac{R^2}{2z^2}$$

$$\Rightarrow V \approx \frac{Q}{4\pi\epsilon_0 z}$$

Point charge!

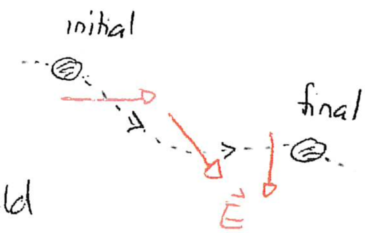
Work and Energy in Electrostatics

How can we reason about electrostatic situations using energy? In general in classical physics, the work-kinetic energy theorem states:

> The change in kinetic energy of any particle satisfies
>
> $$\Delta K = W_{net}$$
>
> where W_{net} = net work done by all forces on particle.

Suppose that we apply this to a test particle with charge Q and the only forces acting on this are electrostatic. Then

$$W = \int_{path} \vec{F} \cdot d\vec{l}$$



But $\vec{F} = Q\vec{E}$ where $\vec{E}$ is the electric field produced by all (other) source charges. So

$$W = Q \int \vec{E} \cdot d\vec{l}$$

$$= Q(-\Delta V)$$

Thus we get

> The work done on a test charge Q is
>
> $$W = -Q\Delta V$$
>
> where $\Delta V = V_f - V_i$ is the change in electrostatic potential from the initial to final location.

We can then define

> The electrostatic potential energy of a point charge Q in the presence of sources that produce potential V is
>
> $$U_{elec} = QV$$

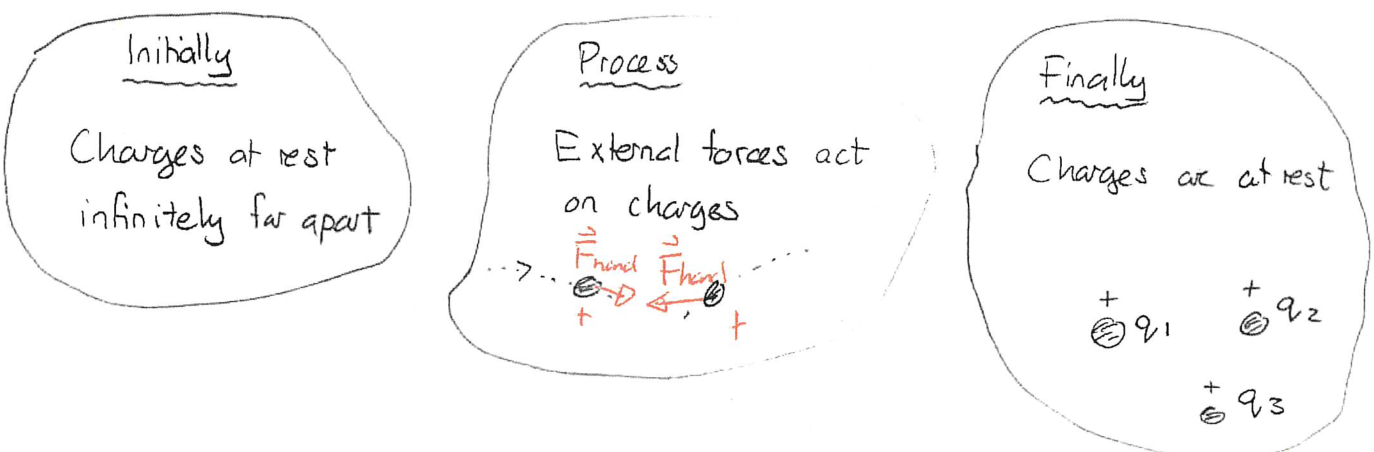
and then

> If a test charge Q is placed in a region where the only forces acting on it are electrostatic then
>
> $$W_{net} = -Q\Delta V$$
>
> and
>
> $$\Delta K + Q\Delta V = 0$$

This is the conservation of energy for electrostatics.

Energy stored in a charge distribution

We can consider the process of assembling a collection of point charges in the following scenario.



We can ask how much work must be done to assemble these charges in this way.

Here

$$W_{net} = W_{electrostatic} + W_{external} = \Delta K = 0$$

$$\Rightarrow \quad W_{external} = - W_{electrostatic}$$

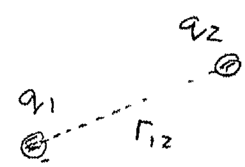
Then we define:

$$\boxed{\begin{array}{l}\text{The energy stored in the charge distribution} \\ \quad = \text{work required to assemble the distribution from} \\ \quad\quad \text{charges initially at rest infinitely far apart.}\end{array}}$$

For a discrete collection of charges $q_1, q_2, \dots$ we proceed as:

1) Determine work to bring charge q_2 into presence of q_1

$$W_{elec} = - q_2 \Delta V \leftarrow \text{produced by } q_1$$



$$= - q_2 \left[V(r_{12}) - V(\infty) \right]$$

$$= - q_2 \left[\frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{12}} - 0 \right]$$

$$= - \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$$

$$\Rightarrow \quad W_{ext} = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$$

2) Determine work to bring q_3 into presence of 1 and 2

3) " " " " q_4 " " " $q_1, q_2,$ and q_3

4) add all external works

Eventually (Section 2.4)

$$W = \frac{1}{2} \sum_{\substack{\text{all charges} \\ j}} q_j \, V(\vec{r}_j)$$

potential from entire distribution

location of q_j

and for continuous distributions

$$\boxed{W = \frac{1}{2} \int_{\text{all space}} \rho(\vec{r}\,') V(\vec{r}\,') \, d\tau'}$$

We can then show:

If an electrostatic charge distribution produces field $\vec{E}$ then the work done to assemble these charges is:

$$W_{ext} = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E} \, d\tau$$

Proof: $\rho(\vec{r}) = \epsilon_0 \vec{\nabla} \cdot \vec{E}$

$$\Rightarrow \quad W = \frac{\epsilon_0}{2} \int (\vec{\nabla} \cdot \vec{E}) \, V \, d\tau$$

But $\vec{\nabla} \cdot (\vec{E} V) = (\vec{\nabla} \cdot \vec{E}) V + \vec{E} \cdot \underbrace{\vec{\nabla} V}_{-\vec{E}} = (\vec{\nabla} \vec{E}) V - \vec{E} \cdot \vec{E}$

Thus

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{\nabla} \cdot (\vec{E} V) \, d\tau + \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E} \, d\tau$$

$$= \frac{\epsilon_0}{2} \underbrace{\overset{0}{\int \cancel{\vec{E} V} \cdot d\vec{a}}}_{\text{infinite boundary}} + \frac{\epsilon_0}{2} \int \vec{E} \, \vec{E} \, d\tau \quad \blacksquare$$

2 Energy stored in a charge distribution

A solid sphere with radius R contains charge that is distributed with density

$$\rho(\mathbf{r}') = \frac{Q}{\pi R^4} r'$$

where Q is the total charge on the sphere.

a) Determine the electric field at all locations.

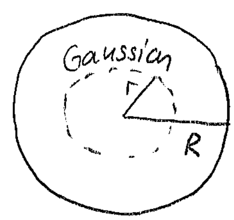
b) Use the electric field to determine the total energy stored in the distribution.

Answer: a) We need the field via Gauss' Law. By symmetry

$$\vec{E} = E_r(r)\hat{r}$$

We use a Gaussian sphere of radius r. Then:

$$\left.\begin{array}{c} r' = r \\ 0 \leq \theta' \leq \pi \\ 0 \leq \phi' \leq 2\pi \end{array}\right\} \quad \begin{aligned} d\vec{a} &= r'^2 \sin\theta' \, d\theta' \, d\phi' \\ &= r^2 \sin\theta' \, d\theta' \, d\phi' \end{aligned}$$



and

$$\vec{E} \cdot d\vec{a} = E_r(r) r^2 \sin\theta' \, d\theta' \, d\phi'$$

So

$$\oint \vec{E} \cdot d\vec{a} = \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \; E_r(r) r^2$$

$$= 4\pi r^2 E_r(r)$$

Then

$$\oint \vec{E} \cdot d\vec{a} = q_{enc}/\epsilon_0 \quad \Rightarrow \quad 4\pi r^2 E_r(r) = q_{enc}/\epsilon_0$$

$$\Rightarrow \quad E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2}$$

Inside ($r<R$) Then q_{enc} is the charge

$$q_{enc} = \int_0^r dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \, r'^2 \sin\theta' \underbrace{\rho(\vec{r}')}_{\frac{Q}{\pi R^4} r'}$$

$$= \frac{Q}{\pi R^4} \underbrace{\int_0^r r'^3 dr'}_{r^4/4} \underbrace{\int_0^\pi \sin\theta' d\theta'}_{2} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} = Q\left(\frac{r}{R}\right)^4$$

Outside ($r>R$) $q_{enc} = Q$.

Thus

$$\vec{E}(\vec{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{Qr^2}{R^4} \hat{r} & r<R \\ \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r} & r>R \end{cases}$$

So $$W_{ext} = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E}\cdot\vec{E}\, d\tau = \frac{\epsilon_0}{2} \int_{\text{inside}} \vec{E}\cdot\vec{E}\, d\tau + \frac{\epsilon_0}{2} \int_{\text{outside}} \vec{E}\cdot\vec{E}\, d\tau$$

Then $$\int_{\text{inside}} \vec{E}\cdot\vec{E}\, d\tau = \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \, r^2 \sin\theta \left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{Qr^2}{R^4}\right)^2$$

$$= \frac{1}{16\pi^2\epsilon_0^2} \frac{Q^2}{R^8} \underbrace{\int_0^R r^6 dr}_{R^7/7} \underbrace{\int_0^\pi \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \frac{1}{4\pi\epsilon_0^2} \frac{Q^2}{7R}$$

Then $\displaystyle\int_{\text{outside}} \vec{E}\cdot\vec{E}\, d\tau = \int_R^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\; r^2 \sin\theta \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{Q^2}{r^4}$

$$= \frac{1}{16\pi^2\epsilon_0^2}\, Q^2 \underbrace{\int_R^\infty \frac{1}{r^2} dr}_{1/R} \underbrace{\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}$$

$$= \frac{1}{4\pi\epsilon_0^2} \frac{Q^2}{R}.$$

Thus

$$W_{ext} = \frac{\epsilon_0}{2} \frac{1}{4\pi\epsilon_0^2} \left\{ \frac{Q^2}{7R} + \frac{Q^2}{R} \right\}$$

$$= \frac{1}{8\pi\epsilon_0} \frac{Q^2}{R} \frac{8}{7}$$

$$\Rightarrow \quad W_{ext} = \frac{1}{7\pi\epsilon_0} \frac{Q^2}{R}$$