Fri: HW 11

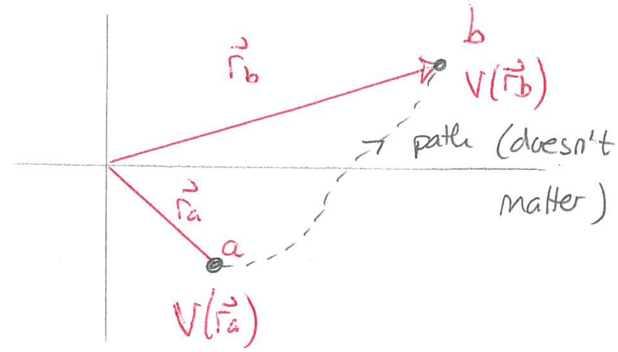
Fri: Read 2.3.4, 2.3.5

## Electrostatic Potential

Electrostatic fields always satisfy $\vec{\nabla}\times\vec{E}=0$ and this eventually implies that the line integral of $\vec{E}$ between any two points is path independent. Thus the difference in electrostatic potential, defined as

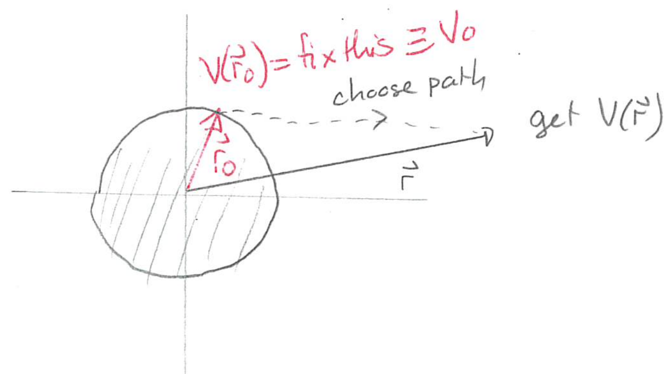
$$\Delta V_{a\to b} = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{l}$$

$$V(\vec{r}_b) - V(\vec{r}_a) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{l}$$




only depends on the two points. This establishes difference in electrostatic potentials. A specific value at each location can be fixed by fixing the potential at one location, denoted $\vec{r}_0$, Then

$$\boxed{V(\vec{r}) = \underbrace{V(\vec{r}_0)}_{\substack{\text{fixed}\\ \equiv V_0}} - \int_{\underbrace{\vec{r}_0}_{\text{choice}}}^{\vec{r}} \vec{E}\cdot d\vec{l}}$$



## 1 Electrostatic potential: uniformly charged cylinder

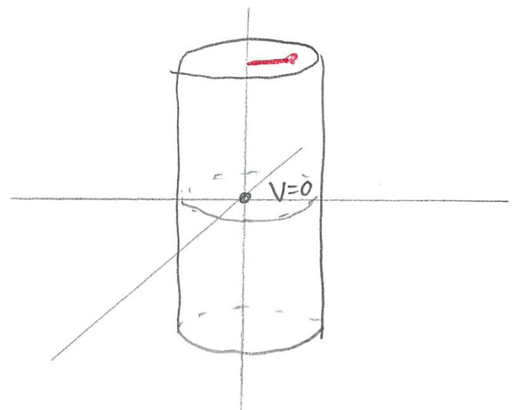
A cylinder with radius $R$ holds a charge with uniform density $\rho$. The electric field (in cylindrical coordinates) is:

$$\mathbf{E} = \begin{cases} \dfrac{1}{2\epsilon_0}\rho s\,\hat{\mathbf{s}} & \text{if } s \leqslant R \\ \dfrac{1}{2\epsilon_0}\rho\dfrac{R^2}{s}\,\hat{\mathbf{s}} & \text{if } s \geqslant R. \end{cases}$$

Suppose that $V = 0$ at the center of the cylinder. Determine the potential at all other points.

Answer: Here $\vec{r}_0 = 0$ and $V(0) = 0$.

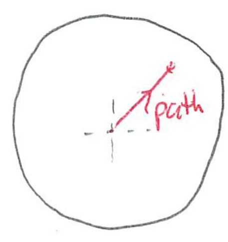
We first establish that the z-axis is an equipotential. Along the axis $\vec{dl} = dz\,\hat{z}$ and $\vec{E}\cdot\vec{dl} = 0$. Thus




$$V\left(\begin{array}{c}\text{any pt on}\\ z \text{ axis}\end{array}\right) = V(0) - 0$$
$$= 0$$

Thus the entire z-axis is an equipotential.

The potential at any other point can be determined by following a path radially outward. Then along the path




$$\left.\begin{array}{l} 0 \leq s' \leq s \\ \phi' = \text{const} \\ z' = \text{const} \end{array}\right\} \quad \vec{dl} = ds'\,\hat{s}$$

Suppose we want the potential at a point inside the cylinder

Then

$$\vec{E}\cdot\vec{dl} = \frac{1}{2\epsilon_0}\rho s'\hat{s}\cdot ds'\hat{s}$$

$$= \frac{1}{2\epsilon_0}\rho s' ds'$$

So
$$\int_0^s \vec{E}\cdot\vec{dl} = \frac{1}{2\epsilon_0}\int_0^s \rho s' ds'$$

$$= \frac{\rho}{4\epsilon_0} s'^2\Big|_0^s = \frac{\rho s^2}{4\epsilon_0}$$

Then
$$\Delta V_{0\to s} = V(s) - V(0) = -\int_0^s \vec{E}\cdot\vec{dl}$$

$$\Rightarrow V(s) - 0 = -\frac{\rho s^2}{4\epsilon_0} \Rightarrow \boxed{V(s) = -\frac{\rho s^2}{4\epsilon_0} \qquad s \leq R}$$

Now outside the cylinder, follow a similar path, but

$$\int \vec{E}\cdot\vec{dl} = \int_0^R \vec{E}\cdot\vec{dl} + \int_0^R \vec{E}\cdot\vec{dl} \quad \text{here } \vec{E}\cdot\vec{dl} = \frac{\rho}{2\epsilon_0}\frac{R^2}{s'}$$

$$= \frac{\rho R^2}{4\epsilon_0} + \int_R^s \frac{\rho}{2\epsilon_0}\frac{R^2}{s'}ds'$$

(prev result with $s=R$)

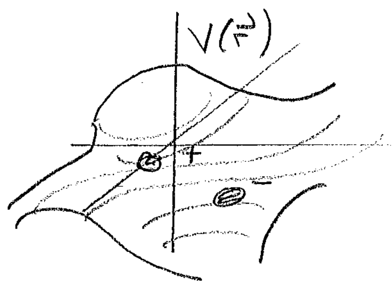
$$= \frac{\rho R^2}{4\epsilon_0} + \frac{\rho R^2}{2\epsilon_0}\ln s'\Big|_R^s = \frac{\rho R^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{s}{R}\right)\right]$$

$$\Rightarrow \boxed{V(s) = -\frac{\rho R^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{s}{R}\right)\right] \qquad s \geq R}$$

Note that the potential

# Electrostatic potential and electric field

The electrostatic potential is constructed from the electric field. Is the reverse process possible. That is, given an electrostatic potential $V(\vec{r})$ produced by source charges, can we obtain the electric field $\vec{E}(\vec{r})$ produced by the same source charges?




Need $\vec{E}(\vec{r})$

By definition the change in electrostatic potential from one location to another is

$$\Delta V \equiv V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

But the fundamental theorem gives:

$$V(\vec{r}) - V(\vec{r}_0) = \int_{\vec{r}_0}^{\vec{r}} \vec{\nabla} V \cdot d\vec{l}$$

$$\Rightarrow \int_{\vec{r}_0}^{\vec{r}} \vec{\nabla} V \cdot d\vec{l} = \int_{\vec{r}_0}^{\vec{r}} -\vec{E} \cdot d\vec{l}$$

This can only be true for all points $\vec{r}$ and $\vec{r}_0$ if

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

**2 Electrostatic potential and field: point charge**

The electrostatic potential for a point charge is:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}.$$

Determine the electric field using this.

Answer:

$$\vec{E} = -\vec{\nabla} V$$

and

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\cancel{\frac{\partial V}{\partial \theta}}^{0}\hat{\theta} + \frac{1}{r\sin\theta}\cancel{\frac{\partial V}{\partial \phi}}^{0}\hat{\phi}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{Q}{r}\right)\hat{r}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

Thus

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

## 3 Electrostatic potential and field: unifromaly charged cylinder

The electrostatic potential for a uniformly charged cylinder of radius $R$ is (in cylindrical coordinates):

$$V = \begin{cases} -\dfrac{\rho}{4\epsilon_0}\, s^2 & \text{if } s \leqslant R \\ -\dfrac{\rho R^2}{4\epsilon_0}\left[1 + 2\ln\left(\dfrac{s}{R}\right)\right] & \text{if } s \geqslant R. \end{cases}$$

Determine the electric field using this.

Answer: $\vec{E} = -\vec{\nabla} V$

In cylindrical co-ordinates

$$\vec{\nabla} V = \frac{\partial V}{\partial s}\hat{s} + \cancel{\frac{1}{s}\frac{\partial V}{\partial \phi}}^{\,0}\hat{\phi} + \cancel{\frac{\partial V}{\partial z}}^{\,0}\hat{z}$$

$$\Rightarrow \vec{E} = -\frac{\partial V}{\partial s}\hat{s}$$

For $s \leq R$ (inside cylinder)

$$\frac{\partial V}{\partial s} = -\frac{2\rho}{4\epsilon_0} s \quad \Rightarrow \quad \boxed{\vec{E} = \frac{\rho s}{2\epsilon_0}\hat{s} \qquad s \leq R}$$

For $s \geq R$ (outside cylinder)

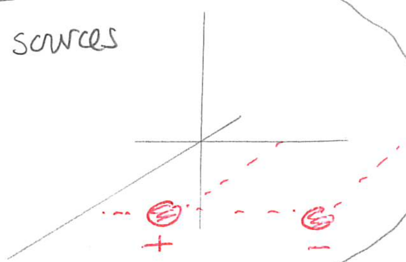
$$\frac{\partial V}{\partial s} = -\frac{\rho R^2}{4\epsilon_0}\, 2 \frac{\partial}{\partial s}\ln\left(\frac{s}{R}\right)$$

$$= \frac{-2\rho R^2}{4\epsilon_0}\,\frac{1}{s}$$

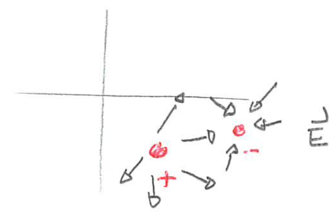
$$\Rightarrow \boxed{\vec{E} = \frac{\rho R^2}{2\epsilon_0 s}\hat{s} \qquad s \geq R}$$

To summarize we have two equivalent formalisms for determining electrostatic fields

sources

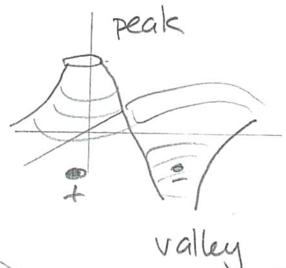



Produce <u>electric field</u>, a <u>vector field</u>




Calculate using Coulomb's Law, Gauss' Law

Produce <u>electrostatic potential</u>, a <u>scalar field</u>




$$\Delta V = V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

any path

$$\vec{E} = -\vec{\nabla} V$$

Note that for the electrostatic potential:

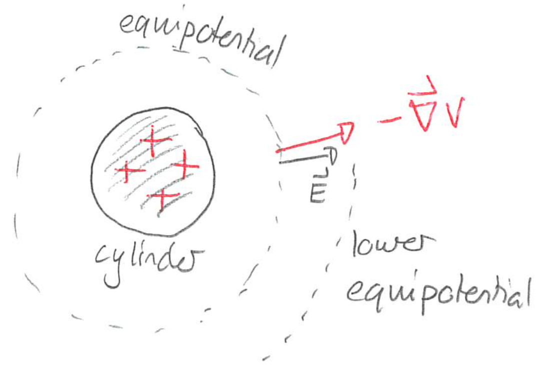
1) the potential is only define via differences in potential between two points. This definition is unambiguous

2) given one possibility for an electrostatic potential $V(\vec{r})$ many other possibilities result in the same electric fields and therefore in the same physics. Thus let

$$\left.\begin{aligned} V_1(\vec{r}) &= V(\vec{r}) + V_1 \leftarrow \text{constant} \\ V_2(\vec{r}) &= V(\vec{r}) + V_2 \end{aligned}\right\} \quad \begin{aligned} \Delta\vec{V}_1 &= \Delta V(\vec{r}) \\ \Delta\vec{V}_2 &= \Delta V(\vec{r}) \end{aligned} \Rightarrow \text{give same fields}$$

3) the electric field is perpendicular to the contours of the electric potential (equipotentials) in the "downhill" direction.

We can illustrate this




Demo: Plots at teaching/topics/elmag/upperdiv/*.jpeg.

* Single pt
* Two pt charges
* Quadrupole

## Poisson's equation

So far we have computed the electrostatic potential via the electric field. Can we obtain the potential without first obtaining the field. This would definitely require the charge distribution. To see this, note that

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

and $\quad \vec{E} = -\vec{\nabla}V \quad$ imply

$$-\vec{\nabla}\cdot\vec{\nabla}V = \frac{\rho}{\epsilon_0}$$

Thus

$$\boxed{\nabla^2 V = -\rho/\epsilon_0 \quad \Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho(x,y,z)}{\epsilon_0}}$$

This is Poisson's equation and it sets up a second order differential equation such that the potential can be determined directly from the charge distribution.