Phys 311 2020

Lecture 16

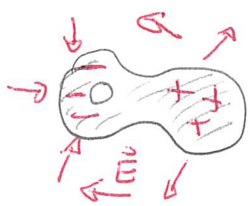
Tues: HW 10

Weds: 2.3.2 → 2.3.4

Electrostatic Field Equations

Starting with the basic force law for electrostatics we arrived at differential relationships that are satisfied by electrostatic electric fields. This results in a differential equation formalism for electrostatics.

Given a stationary source charge distribution $\rho(\vec{r})$. This produces electric field $\vec{E}$



→ The electric field has to satisfy

$$\vec{\nabla}\cdot\vec{E} = \rho(\vec{r})/\epsilon_0$$

$$\vec{\nabla}\times\vec{E} = 0$$

For example using Cartesian co-ordinates the electric field

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

satisfies

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho(x,y,z)}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0$$

$$\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = 0$$

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0$$

In this sense the mathematics of electrostatic fields has become a question of satisfying a set of coupled differential equations.

Then the Helmholtz theorem guarantees (by explicit construction) that there exists a unique solution to these equations that approaches $\vec{E} \to 0$ at infinite distances from sources.

The task of solving these equations analytically can be very complicated or apparently impossible. Some of the difficulties arise with typical issues of solving differential equations. There are also difficulties associated with discontinuities in source charges.

We so far have two techniques that improve prospects of solving these analytically.

1) for a point charge at $\vec{r}\,'$ the field at $\vec{r}$ is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$$ Coulomb's Law

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$. This implies that for continuous distributions

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$$

2) for any distribution and any closed surface, S

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} \underbrace{\int \rho(\vec{r}\,')d\tau'}_{\text{enclosed region}} = q_{enc}/\epsilon_0$$

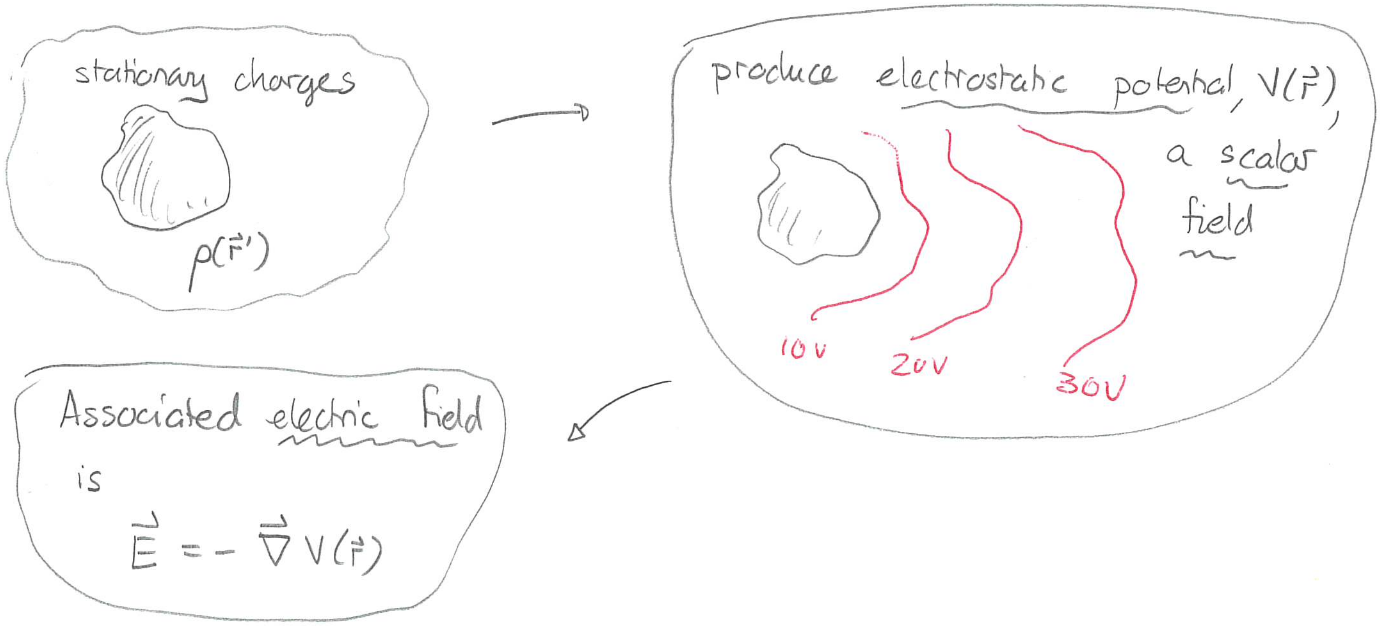
This is Gauss' Law and is useful in situations of high symmetry.

Electrostatic Potential

We will now provide an alternative formulation of electrostatics that uses an entity called the electrostatic potential. The potential formulation will be exactly equivalent to the field formulation but it will offer:

1) easier techniques for computing certain fields, especially in regions with no charge
2) energy conservation in electrostatics
3) a step toward a complete reformulation of all electromagnetism in terms of potentials. This will be essential for computing the electric and magnetic fields produced by moving point charges.

The eventual framework will be:



We need to establish

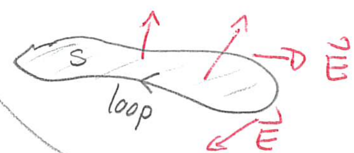
* the existence of such a scheme
* a method for computing potential from charge density.

The essential idea is that $\vec{\nabla}\times\vec{E}=0$ and also $\vec{\nabla}\times\vec{\nabla}f=0$ for any scalar function f. Thus it is possible that there is some function $f(\vec{r})$ s.t. $\vec{E}=\vec{\nabla}f$. We can verify this via an explicit construction.

Note $\vec{\nabla}\times\vec{E}=0 \Rightarrow$ Stoke's Theorem

and any loop

$$\int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a} = \oint_{\text{loop}} \vec{E}\cdot d\vec{l} = 0$$

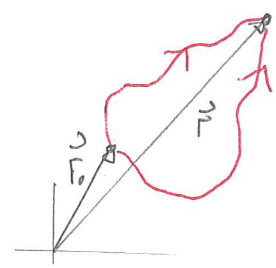


Around any closed loop

$$\oint \vec{E}\cdot d\vec{l} = 0$$

$$\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$$ line integral

is independent of path from $\vec{r}_0$ to $\vec{r}$



Define electric / electrostatic potential via a difference / change

$$V(\vec{r}) = V(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$$ line integral

$\hookrightarrow$ show $\vec{\nabla}V = -\vec{E} \Leftrightarrow \vec{E} = -\vec{\nabla}V$

To demonstrate this we note that Stokes' theorem implies that

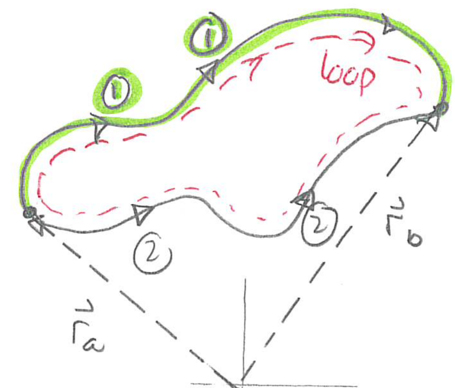
> If $\vec{E}$ is an electrostatic field then, around any closed loop
>
> $$\oint_{\text{loop}} \vec{E}\cdot d\vec{l} = 0$$

The next step is to demonstrate the path independence of a line integral of an electrostatic field. Specifically.

> Given two points described by position vectors $\vec{r}_a$ and $\vec{r}_b$
>
> $$\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{l}$$
>
> is independent of the path between the points.

Proof: Consider two points at $\vec{r}_a$ and $\vec{r}_b$

Then:



$$0 = \oint_{\text{loop}} \vec{E}\cdot d\vec{l} = \underset{\substack{1\\ \text{forwards}}}{\int_{\vec{r}_a}^{\vec{r}_b}} \vec{E}\cdot d\vec{l} + \underset{\substack{2\\ \text{backwards}}}{\int_{\vec{r}_a}^{\vec{r}_b}} \vec{E}\cdot d\vec{l} = \underset{\substack{1\\ \text{forwards}}}{\int_{\vec{r}_a}^{\vec{r}_b}} \vec{E}\cdot d\vec{l} - \underset{\substack{2\\ \text{forwards}}}{\int_{\vec{r}_a}^{\vec{r}_b}} \vec{E}\cdot d\vec{l}$$

$$\Rightarrow \underset{2}{\int_{\vec{r}_a}^{\vec{r}_b}} \vec{E}\cdot d\vec{l} = \underset{1}{\int_{\vec{r}_a}^{\vec{r}_b}} \vec{E}\cdot d\vec{l}$$

This is path independence ∎

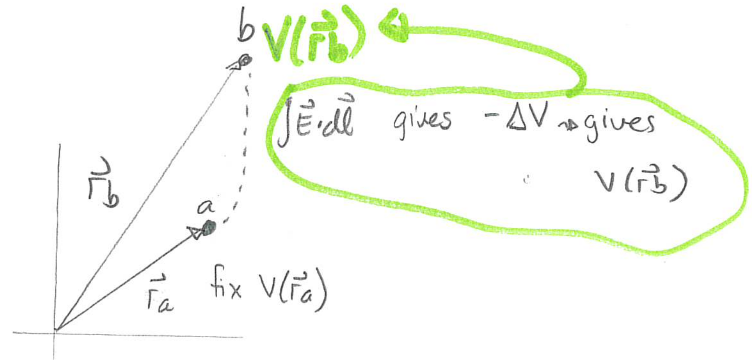
We can use this to establish

1) a scalar function, $V(\vec{r})$, with units of Volts

2) the scalar function cannot be defined absolutely uniquely but the difference between the scalar function at any two points can be defined to be:

$$\boxed{\Delta V_{a \cdot \text{to}\, b} = V(\vec{r}_b) - V(\vec{r}_a) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot \vec{dl}}$$

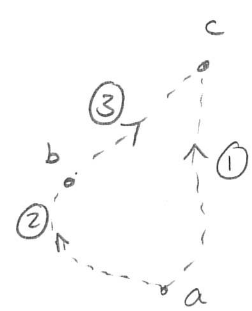
We can establish unique differences from one location to another. We cannot unambiguously determine a value at any location. If we were to arbitrarily fix a value at one location we could map values for the potential elsewhere.



Note that this only makes sense because the line integral of the electric field is path independent. So

$$\underset{①}{\Delta V_{a\ \text{to}\ c}} = \underset{\text{via } ②+③}{\Delta V_{a\ \text{to}\ c}}$$

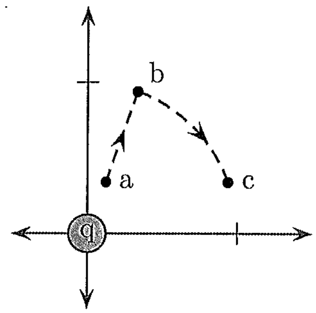
$$= \Delta V_{a \to b} + \Delta V_{b \to c}$$



1 Electrostatic potential: point charge

A point particle with charge q is located at the origin and produces electric field

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q\,\hat{\mathbf{r}}}{r^2}.$$



a) Determine ΔV from point a to b using a line integral along the illustrated path.

b) Determine ΔV from point b to c using a line integral along the illustrated path.

Answer: a) Along the radial path in spherical co-ordinates

$$\left.\begin{array}{l} r_a \le r \le r_b \\ \theta, \phi \text{ fixed} \end{array}\right\} \quad \vec{dl} = dr\,\hat{r}$$

So

$$\vec{E}\cdot\vec{dl} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr$$

$$\Rightarrow \quad \Delta V_{a\to b} = -\int \vec{E}\cdot\vec{dl} = -\frac{1}{4\pi\epsilon_0}\,q\int_{r_a}^{r_b}\frac{dr}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a}\right]$$

$$\Rightarrow \quad \Delta V_{a\to b} = \frac{q}{4\pi\epsilon_0 r_b} - \frac{q}{4\pi\epsilon_0 r_a}$$

b) Along this path

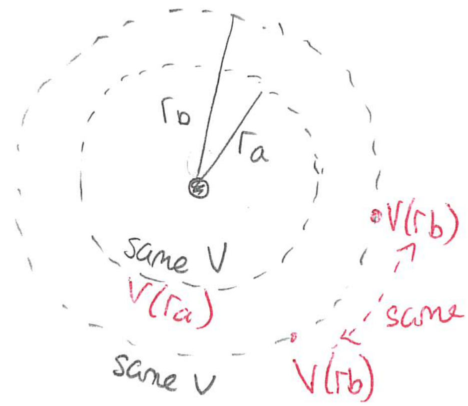
$$\left.\begin{array}{l} r = r_b \\ \phi_a \le \phi \le \phi_c \\ \theta = \text{const} \end{array}\right\} \quad \vec{dl} = r\,d\phi\,\hat{\phi} = r_b\,d\phi\,\hat{\phi}$$

Now $\quad \vec{E}\cdot d\vec{l} = 0 \quad \Rightarrow \quad \int_{\vec{r}_b}^{\vec{r}_c} \vec{E}\cdot d\vec{l} = 0$

$$\Rightarrow \quad \Delta V_{b\to c} = 0. \quad \Rightarrow \quad V(\vec{r}_c) = V(\vec{r}_b) \quad \blacksquare$$

We see that we can map a system of equipotentials. In order to fix the potential at all locations we need only fix it at one point.

Thus we can do:



1) choose a reference pt (or a reference contour), $\vec{r}_0$

2) Set $V(\vec{r}_0)$ to a define value.

3) At any other point $\vec{r}$, follow any line integral from $\vec{r}_0 \to \vec{r}$

Then

$$\Delta V_{\vec{r}_0 \to \vec{r}} = V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l} \quad \Rightarrow \quad \boxed{V(\vec{r}) = \underset{\text{fixed}}{V(\vec{r}_0)} - \int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}}$$

Example: With the point charge, set $\vec{r}_0 \to \infty$ and $V(\vec{r}_0) = 0$. Then

$$\underbrace{\Delta V_{r_0 \text{ to } r}} = \frac{q}{4\pi\epsilon_0 r} - \underbrace{\frac{q}{4\pi\epsilon_0 r_0}}_{\to 0}$$

$$V(r) - \underbrace{\cancel{V(r_0)}}_{0} = \frac{q}{4\pi\epsilon_0 r}$$

$$\Rightarrow \quad V(r) = \frac{q}{4\pi\epsilon_0 r}$$

Demo PhET Charges + fields