Fri: HW by 5pm

Mon: Read 2.3.1 → 2.3.3

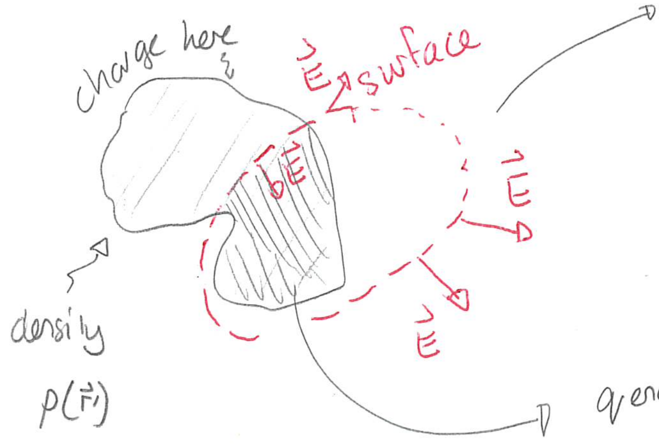
Tues: HW by 5pm (longer!)

Gauss' Law

Gauss' Law relates the flux of an electric field through any closed surface to the charge enclosed by the surface.

> Let S be any closed surface. Let $\vec{E}$ be the electric field produced by a stationary charge distribution. Then:
>
> $$\oint_S \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$$
>
> where q_{enc} is the total charge enclosed by the surface

This is illustrated as:



compute $\oint \vec{E}\cdot d\vec{a}$ by integrating over surface

q_{enc} is charge in this region (intersection)

$$q_{enc} = \int_{\text{portion of distrib in S}} \rho(\vec{r}')\, d\tau'$$

In general Gauss' Law can be used to easily determine fields for highly symmetric situations. The procedure is:

Constrain field by symmetry arguments $\longrightarrow$ e.g. spherically symmetric charge distrib $\rho(\vec{r}') = \rho(r')$

$$\vec{E} = E_r(r,\theta,\phi)\hat{r} + \cancel{E_\theta(r,\theta,\phi)}\hat{\theta} + \cancel{E_\phi(r,\theta,\phi)}\hat{\phi}$$

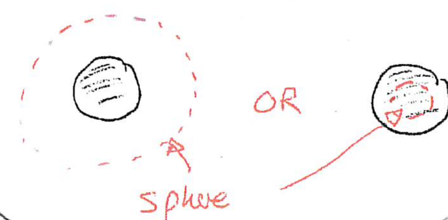
only depends on r

$\Downarrow$

$$\vec{E} = E_r(r)\hat{r}$$

Choose Gaussian surface that respects symmetry e.g. on which $\vec{E}$ is constant $\longrightarrow$ e.g. spherical symmetry

Gaussian surface does not need to coincide with physical surface $\longrightarrow$



Do surface integral $\longrightarrow$ e.g. spherical symmetry

$$\oint \vec{E} \cdot d\vec{a} = 4\pi r^2 E_r(r)$$

Evaluate q_{enc} $\longrightarrow$ $q_{enc} = \int \rho(\vec{r}')\,d\tau'$ inside Gaussian surface

Connect $\longrightarrow$

$$4\pi r^2 E_r(r) = \frac{q_{enc}}{\epsilon_0}$$

$$E_r(r) = \frac{1}{4\pi\epsilon_0}\frac{q_{enc}}{r^2}$$

1 Field produced by an infinite charged sheet

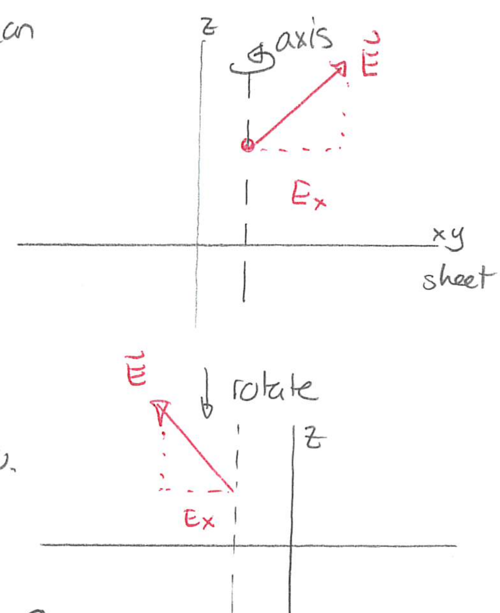
An infinitesimally thin sheet lies in the xy plane and is positively charged with a uniform surface charge density σ.

a) Use symmetry arguments to simplify the general form of the electric field

$$\mathbf{E} = E_x(x,y,z)\hat{\mathbf{x}} + E_y(x,y,z)\hat{\mathbf{y}} + E_z(x,y,z)\hat{\mathbf{z}}.$$

b) Use Gauss' Law to determine the electric field at any point beyond the sheet.

Answer: a) Suppose that $E_x \neq 0$. We can rotate through 180° about an axis through the field point. Then E_x will be reversed. But the charge distribution is unaltered. This is only possible if $E_x = 0$.



Thus $E_x = 0$.

A similar argument gives $E_y = 0$.

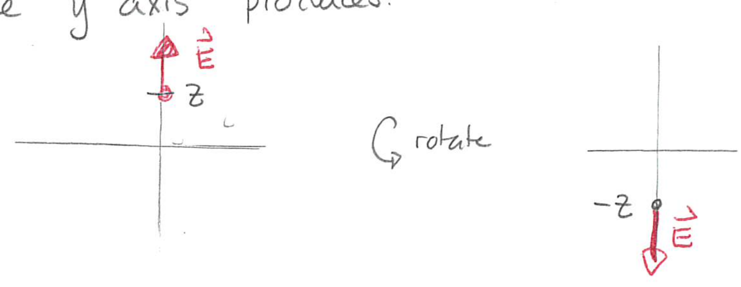
Thus

$$\vec{E} = E_z(x,y,z)\,\hat{z}$$

Symmetry implies that this cannot depend on x, y. Thus

$$\vec{E} = E_z(z)\,\hat{z}$$

Now there is one more crucial symmetry. Rotating about the y axis produces:

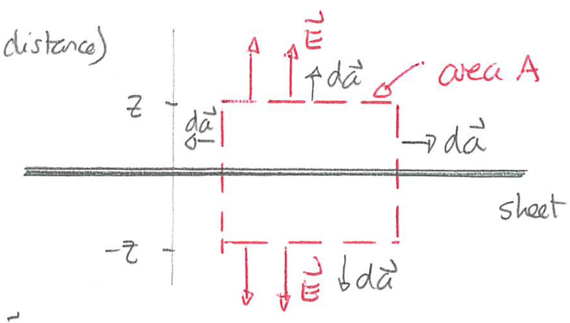


Thus

$$E_z(-z) = -E_z(z)$$

b) The surface is a "pill box" that spans the sheet. It has

* top + bottom parallel to sheet (same distance)
* sides perpendicular to sheet.



Thus

$$\oint \vec{E}\cdot d\vec{a} = \int_{top} \vec{E}\cdot d\vec{a} + \int_{bottom} \vec{E}\cdot d\vec{a} + \int_{sides} \vec{E}\cdot d\vec{a}$$

We do one at a time:

* sides $\quad \vec{E}\cdot d\vec{a} = 0 \quad \Rightarrow \quad \int_{sides} \vec{E}\cdot d\vec{a} = 0$

* top: $\quad d\vec{a} = dx\,dy\,\hat{z} \quad \Rightarrow \quad \vec{E}\cdot d\vec{a} = E_z(z)\,dx\,dy$

$$\Rightarrow \int_{top} \vec{E}\cdot d\vec{a} = \iint_{top} E_z(z)\,dx\,dy = E_z(z) \underbrace{\iint_{top} dx\,dy}_{\text{area } A}$$

$$\Rightarrow \int_{top} \vec{E}\cdot d\vec{a} = E_z(z)\,A$$

* bottom: $\quad \left.\begin{array}{l} d\vec{a} = -dx\,dy\,\hat{z} \\ \vec{E} = E_z(-z)\,\hat{z} \end{array}\right\} \Rightarrow \vec{E}\cdot d\vec{a} = -E_z(-z)\,dx\,dy$

$$= E_z(z)\,dx\,dy.$$

so $$\int_{bottom} \vec{E}\cdot d\vec{a} = E_z(z) \iint_{bottom} dx\,dy = E_z(z)\,A.$$

Thus $$\oint \vec{E}\cdot d\vec{a} = 2A\,E_z(z)$$

Now Gauss' Law gives:

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0} \Rightarrow 2A\, E_z(z) = \frac{q_{enc}}{\epsilon_0}$$

$$\Rightarrow E_z(z) = \frac{q_{enc}}{2\epsilon_0 A}$$

Then the charge enclosed by the Gaussian surface is

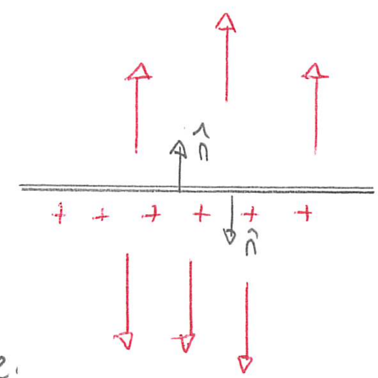
$$q_{enc} = \sigma A$$

Thus

$$E_z(z) = \frac{\sigma A}{2\epsilon_0 A} = \frac{\sigma}{2\epsilon_0}$$

We then get

$$\vec{E} = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{z} & z > 0 \\ -\frac{\sigma}{2\epsilon_0}\hat{z} & z < 0 \end{cases}$$



or

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

where $\hat{n}$ is the normal away from the surface.

We see that:

For a uniformly charged infinite sheet the electric field on either side is uniform. It does not depend on the distance from the sheet

Differential form of Gauss' Law

In general for <u>any</u> closed surface

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

$$= \frac{1}{\epsilon} \int_{\text{region inside } S} \rho(\vec{r}')\, d\tau'$$

Now the divergence theorem gives:

$$\oint_S \vec{E} \cdot d\vec{a} = \int_{\text{region inside } S} \vec{\nabla} \cdot \vec{E}\, d\tau$$

$$\Rightarrow \int_{\text{region inside } S} \vec{\nabla} \cdot \vec{E}\, d\tau = \int_{\text{region inside}} \frac{\rho(\vec{r})}{\epsilon_0}\, d\tau$$

This can only be true for <u>all</u> regions if the integrands are identical. Thus

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}}$$

This is the differential form of Gauss' Law. The argument is reversible. Thus

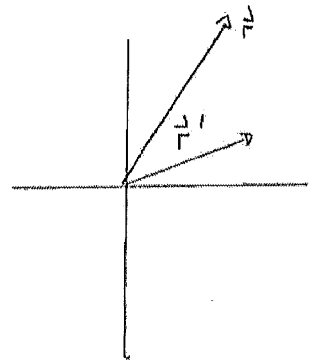
$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0} \Leftrightarrow \oint_S \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}}$$

Coulomb's Law from Gauss' Law

We can show that Coulomb's Law can be derived from Gauss' law. This requires a three dimensional version of the delta function. The three dimensional delta function is defined as follows.

Choose any location $\vec{r}'$. Then for all $\vec{r}$

$$\delta^3(\vec{r}-\vec{r}') = \begin{cases} 0 & \vec{r} \neq \vec{r}' \\ \infty & \vec{r} = r' \end{cases}$$



and

$$\int_{\text{all space}} f(\vec{r})\,\delta^3(\vec{r}-\vec{r}')\,d\tau = f(\vec{r}')$$

Now a point charge at location $\vec{r}'$ can be described by density

$$\rho(\vec{r}) = Q\,\delta^3(\vec{r}-\vec{r}') = \begin{cases} 0 & \vec{r} \neq \vec{r}' \\ \infty & \text{otherwise.} \end{cases}$$

So

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$= \frac{Q}{\epsilon_0}\,\delta^3(\vec{r}-\vec{r}')$$

Now it turns out that

$$\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\,\delta^3(\vec{r}-\vec{r}')$$

Thus:

$$\vec{\nabla}\cdot\vec{E} = \frac{Q}{\epsilon_0}\frac{1}{4\pi}\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \vec{\nabla}\cdot\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{\mathscr{r}^2}\hat{\mathscr{r}}\right)$$

and so $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{\mathscr{r}^2}\hat{\mathscr{r}}$. which is Coulomb's Law.

Curl of $\vec{E}$

Starting with Coulomb's Law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$$

we get

$$\vec{\nabla}\times\vec{E} = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}\,')\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

diff w.r.t $\vec{r}$

But $\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$. Thus

> For any electrostatic field
>
> $$\vec{\nabla}\times\vec{E} = 0$$