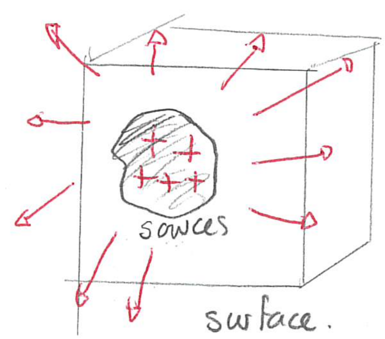
Fri: HW by 5pm

Read 2.2.3 → 2.2.4

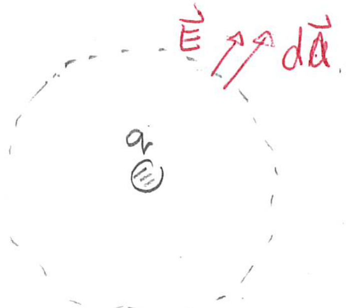
Gauss' Law

Consider a collection of source charges that produce an electric field $\vec{E}$. We can construct an imaginary closed surface and compute the flux through the surface.



$$\boxed{\vec{E} \text{ field flux} = \oint_{\text{surface}} \vec{E} \cdot d\vec{a}}$$

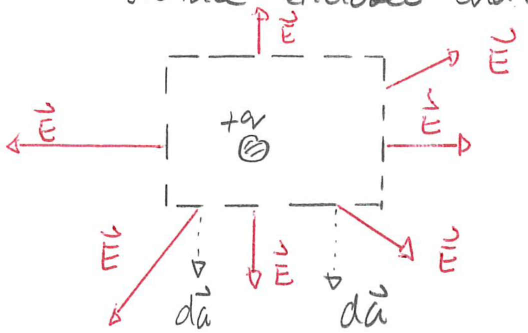
In the special case of a point source charge we could use a spherical surface centered on the point charge and then found that



$$\oint \vec{E} \cdot d\vec{a} = q/\epsilon_0$$

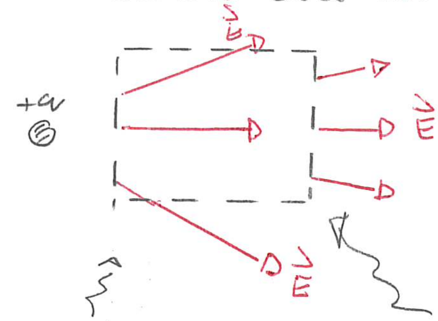
We could repeat this for any type of closed surface in the vicinity of the source. Remarkably the result is always simple.

surface encloses charge



everywhere $\vec{E}\cdot d\vec{a} > 0$

surface does not enclose charge.



This side $\vec{E}\cdot d\vec{a} < 0$

This side $\vec{E}\cdot d\vec{a} > 0$

Add to zero?

For such point charges, we can use: divergence theorem, Coulomb's Law and a distribution described by a delta function to prove:

> Let $\vec{E}$ be the electric field produced by a single point source charge, q. Let S be any closed surface. Then
>
> $$\oint \vec{E}\cdot d\vec{a} = \begin{cases} q/\epsilon_0 & \text{if source is inside } S \\ 0 & \text{if source is outside } S \end{cases}$$

This can be extended to any charge distribution by using the superposition principle. The result is Gauss' Law:

> Let $\vec{E}$ be the electric field produced by a stationary source charge distribution. Let S be any closed surface. Then
>
> $$\oint_S \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$$
>
> where q_{enc} is the total charge enclosed within S.

Gauss' Law can be used to:

1) easily compute electric fields in highly symmetric situations
2) arrive at an electrostatic version of one of Maxwell's equations.

Proof of Gauss' Law

The proof of Gauss' Law uses the three dimensional Dirac delta function:

$$\delta^3(\vec{r}) = \begin{cases} 0 & \vec{r} \neq 0 \\ \infty & \vec{r} = 0 \end{cases}$$

and

$$\int_{\text{all space}} F(\vec{r}\,')\,\delta^3(\vec{r}\,')\,d\tau' = F(\vec{0})$$

Then one can show

$$\vec{\nabla} \cdot \frac{\hat{\imath}}{\imath^2} = 4\pi\,\delta(\vec{\imath})$$

where $\vec{\imath} = \vec{r} - \vec{r}\,'$ and the derivative is w.r.t. $\vec{r} = (x, y, z)$. Now

$$\oint_S \vec{E} \cdot d\vec{a} = \int_{\text{region}} \vec{\nabla} \cdot \vec{E}\,d\tau$$

Then Coulomb's Law gives that

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{\imath^2}\,\hat{\imath}\,d\tau'$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int \vec{\nabla} \cdot \left(\rho(\vec{r}\,')\,\frac{\hat{\imath}}{\imath^2}\right) d\tau'$$

But

$$\vec{\nabla} \cdot \left(\rho(\vec{r}\,')\frac{\hat{\imath}}{\imath^2}\right) = \rho(\vec{r}\,')\,\vec{\nabla} \cdot \frac{\hat{\imath}}{\imath^2}$$

$$= 4\pi\,\rho(\vec{r}\,')\,\delta^3(\vec{\imath})$$

So

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\int \rho(\vec{r}\,')\,\underbrace{\delta^3(\vec{\mathscr{r}})}_{\delta^3(\vec{r}-\vec{r}\,')}\,d\tau'$$

$$= \frac{1}{\epsilon_0}\rho(\vec{r})$$

Thus

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\underbrace{\int \rho(\vec{r})\,d\tau}_{= q_{enc.}}$$

$$= \frac{q_{enc}}{\epsilon_0}. \qquad \blacksquare$$

1 Field due to a charged spherical shell

An infinitesimally thin shell of radius R carries a uniformly distributed charge Q. Determine the electric field at any point inside or beyond the sphere.

Answer: There are two parts:

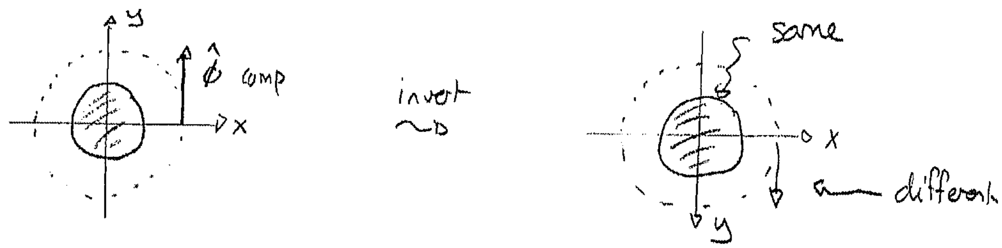
1) a symmetry argument that restricts the form of $\vec{E}$

2) a choice of symmetric integration surface (called a Gaussian surface) followed by integration.

a) Part 1. Field form. In general, in spherical co-ordinates

$$\vec{E} = E_r(r,\theta,\phi)\,\hat{r} + E_\theta(r,\theta,\phi)\,\hat{\theta} + E_\phi(r,\theta,\phi)\,\hat{\phi}$$

a) Can $\vec{E}$ have a $\hat{\phi}$ component? If it did then inverting the sphere about say the x axis would invert this component. But the distribution would remain unaltered. This is impossible.



So $E_\phi = 0$.

b) Can $\vec{E}$ have a $\hat{\theta}$ component? A similar argument says No! So

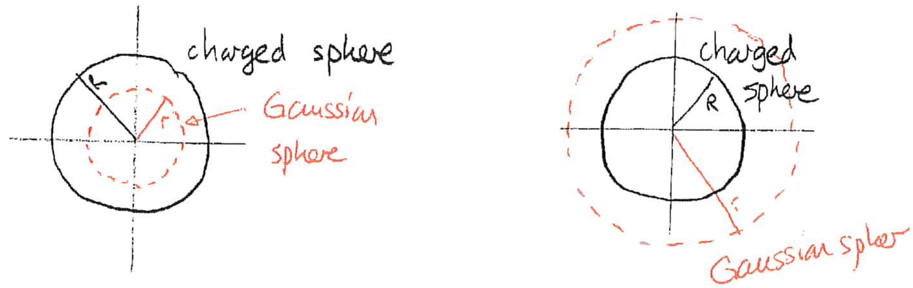
$E_\theta = 0$

c) Then $\vec{E} = E_r(r,\theta,\phi)\,\hat{r}$. Symmetry implies that E_r cannot depend on θ, ϕ. So

$$\vec{E} = E_r(r)\,\hat{r}$$

b) <u>Part 2:</u> Choose the following surface

c) - a sphere of radius r centered at the origin. It does not have to coincide with the physical charged sphere. It could be inside or outside



- on this sphere

$$r = \text{constant}$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi \leq 2\pi$$

$$d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$\Rightarrow \vec{E} \cdot d\vec{a} = E_r(r)\, r^2 \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow \oint_S \vec{E} \cdot d\vec{a} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \underbrace{r^2 E_r(r)}_{\text{does not depend on } \theta, \phi} \sin\theta$$

$$= r^2 E_r(r) \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta}_{4\pi}$$

$$\Rightarrow \oint_S \vec{E} \cdot d\vec{a} = 4\pi r^2 E_r(r)$$

By Gauss' Law

$$4\pi r^2 E_r(r) = \frac{q_{enc}}{\epsilon_0}$$

$$\Rightarrow E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2}$$

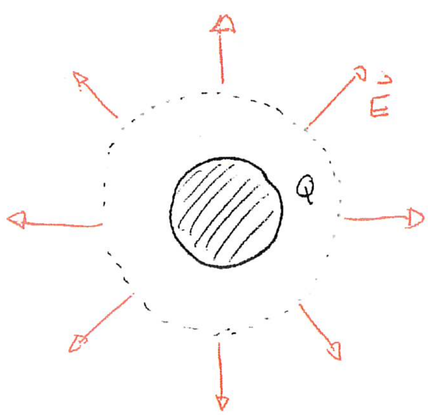
Now if $r > R$ $\quad q_{enc} = Q$

$r < R$ $\quad q_{enc} = 0.$

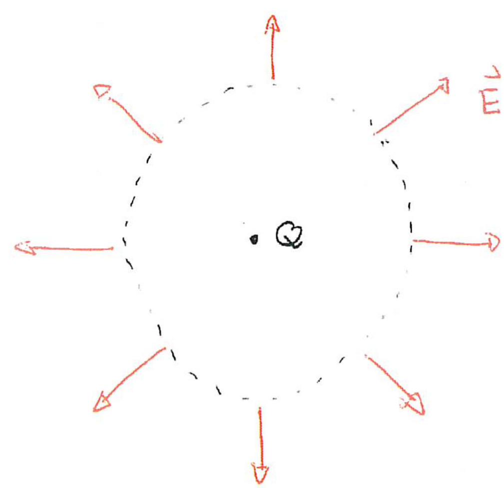
So
$$E_r(r) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2} & r > R \quad \text{outside} \\ 0 & r < R \quad \text{inside} \end{cases}$$

$$\Rightarrow \quad \vec{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\hat{r} & \text{if } r > R \quad \text{(outside)} \\ 0 & \text{if } r < R \quad \text{(inside)} \end{cases}$$

Note that for any such spherical arrangement



same as



There are various important points regarding the use of Gauss' Law

1) there are usually two surfaces:

   * a physical surface inside of which or on which the charge resides

   * a Gaussian surface which does not have to coincide with the physical surface

2) The mechanism will only allow you to calculate the field at the Gaussian surface. Therefore the Gaussian surface must be allowed to vary.

3) The method requires a high degree of symmetry and the field component must not vary along the Gaussian surface in order for the method to succeed. In the example we had

$$\vec{E} = \underbrace{E_r(r)}_{\text{does not vary on a Gaussian sphere..}} \hat{r}$$

4) One might need to integrate to determine the enclosed charge:

$$q_{enc} = \int_{\text{inside Gaussian surface}} \rho(\vec{r}')\, d\tau'$$

2 Uniformly charged cylinder

An infinitely long cylinder of radius R contains a uniform volume charge density ρ.

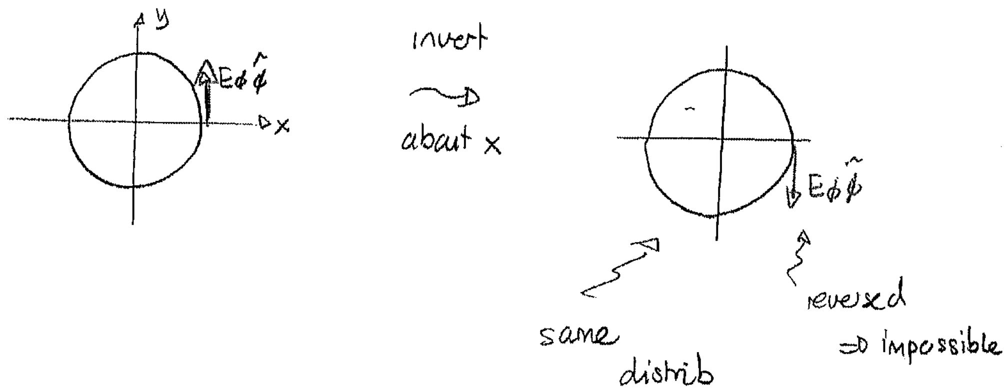
a) Use symmetry arguments to simplify the general form of the electric field

$$\mathbf{E} = E_s(s,\phi,z)\hat{\mathbf{s}} + E_\phi(s,\phi,z)\hat{\boldsymbol{\phi}} + E_z(s,\phi,z)\hat{\mathbf{z}}.$$

b) Use Gauss' Law to determine the electric field at any point beyond the cylinder.

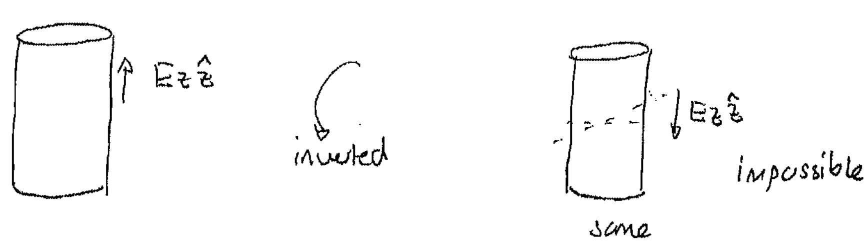
c) Use Gauss' Law to determine the electric field at any point within the cylinder.

Answer: a) Suppose that $E_\phi \neq 0$ Then along the axis



Thus $E_\phi = 0$.

Suppose $E_z \neq 0$ then



Thus $E_z = 0$

So $\vec{E} = E_s(s,\phi,z)\,\hat{s}$. But $\vec{E}$ cannot depend on ϕ or z.

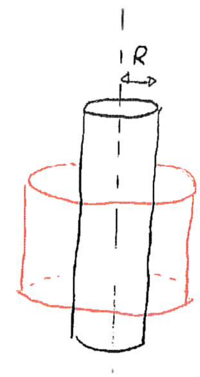
So

$$\vec{E} = E_s(s)\,\hat{s}$$

b) Gaussian surface:

cylinder along z axis with height h

radius of cylinder $= s$. Three surfaces:



top $d\vec{a} = s' ds' d\phi' \hat{z}$

bottom $d\vec{a} = -s' ds' d\phi' \hat{z}$

side $d\vec{a} = s' d\phi' dz' \hat{s}$

$s' = s$

$0 \le \phi' \le 2\pi$

$0 \le z' \le h$

Thus

$$\oint \vec{E} \cdot d\vec{a} = \int_{top} \vec{E} \cdot d\vec{a} + \int_{bottom} \vec{E} \cdot d\vec{a} + \int_{side} \vec{E}\, d\vec{a}.$$

On top/bottom $\vec{E} \cdot d\vec{a} = 0$. On side

$$\int \vec{E} \cdot d\vec{a} = \int_0^h \int_0^{2\pi} E_s(s)\, s\, d\phi' dz' = 2\pi h s\, E_s(s).$$

Thus

$$2\pi h s\, E_s(s) = \frac{q_{enc}}{\epsilon_0} \Rightarrow E_s(s) = \frac{1}{2\pi\epsilon_0} \frac{1}{h} \frac{q_{enc}}{s}.$$

We need the charge enclosed within the Gaussian surface

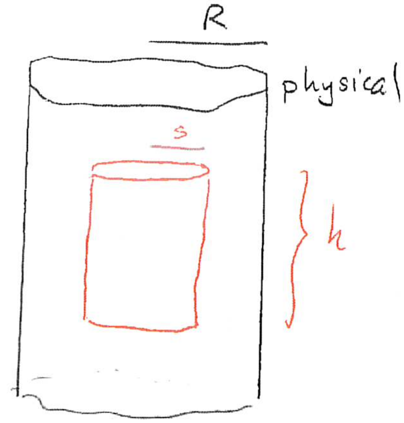
$q_{enc} = \rho \times$ volume of cylinder inside Gaussian surface

$= \rho \pi R^2 h$

Thus $$E_s = \frac{1}{2\cancel{\pi}\epsilon_0} \frac{1}{\cancel{h}} \frac{\rho \cancel{\pi} R^2 \cancel{h}}{s} = \frac{\rho R^2}{2\epsilon_0 s}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\rho R^2}{2\epsilon_0 s} \hat{s} \qquad s \ge R}$$

b) Use a Gaussian surface inside.



Again

$$\oint_S \vec{E} \cdot d\vec{a} = 2\pi h s \, E_s(s)$$

Now the enclosed charge is:

$$q_{enc} = \rho \times \underbrace{\text{vol Gaussian surface}}_{\pi s^2 h}$$

$$= \rho \pi s^2 h$$

$$\Rightarrow \quad 2\cancel{\pi} \cancel{h} \cancel{s} \, E_s(s) = \rho \pi s^{\cancel{2}} \cancel{h} / \epsilon_0$$

$$\Rightarrow \quad E_s(s) = \frac{\rho \pi s}{2\epsilon_0}$$

$$\Rightarrow \quad \boxed{\vec{E} = \frac{\rho s}{2\epsilon_0} \hat{s} \qquad s \leq R}$$