Phys 311 2020

Lecture 13

Tues: HW 8

Weds: Read 2.2.1 → 2.2.3

## Electric field for continuous distributions: two dimensions

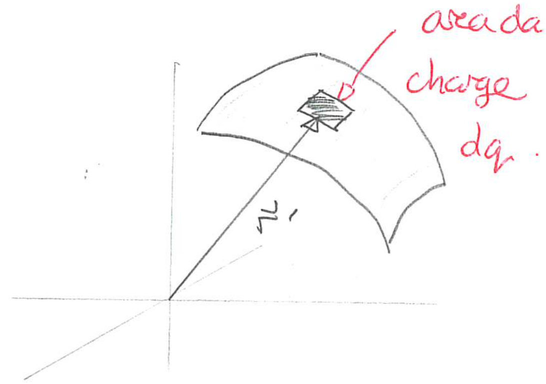
In general charge can be distributed on two or three dimensional objects. The same decomposition and integration scheme will allow us to determine the electric field produced by such distributions.

In two dimensions the charge distribution can be described by a surface charge density, $\sigma(\vec{r})$, which is a scalar function with units $C/m^2$. This is defined so that

The charge in the segment with area $da'$ located at $\vec{r}'$ is

$$dq = \sigma(\vec{r}')\, da'$$

depends on coords: $x', y', z'$ or $s', \phi', z' \ldots$



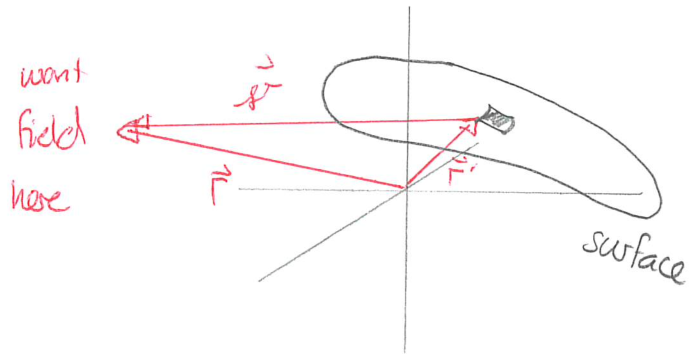

The charge in any extended region is

$$\boxed{\text{charge} = \int_{\text{region}} \sigma(\vec{r}')\, da'}$$

→ This is a double integral and must be constructed by decomposition

CANNOT integrate w.r.t variable "a"!

Now the process of computing the electric field is to decompose the distribution as illustrated. Then the illustrated portion contributes




$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\sigma(\vec{r}')\,da'}{\mathscr{r}^2}\,\hat{\mathscr{r}}$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$\vec{r}$ = location of field point

$\vec{r}'$ = " " source region.

The total field is:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\substack{\text{charge}\\ \text{surface}}} \frac{\sigma(\vec{r}')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,da'$$

contains variables e.g. $x', y'$

depends on $x', y'$

range over all $x', y'$ on surface

By setting $\sigma(\vec{r}') = 0$ outside the charge distribution we can integrate over all space

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\sigma(\vec{r}')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,da'$$

## 1 Field due to a disk of charge

A disk of radius $R$ lies in the $xy$ plane with its center at the origin. The surface charge density on the disk is

$$\sigma(\mathbf{r}') = \alpha \frac{\cos\phi'}{s'}$$

where $\phi'$ is the angle counterclockwise from the $+x$ axis, $s'$ is the distance from the center and $\alpha$ is a constant with units of C/m.

a) Sketch the charge density on the disk qualitatively.

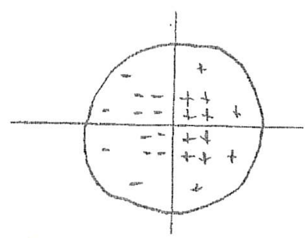
b) Determine an expression for the total charge, $Q$ on the disk. Is $Q = \sigma A$ where $A$ is the area of the disk?

c) Determine an expression for the electric field at any point along the $z$ axis.

d) To verify that your expression is plausible, check what it gives at the center of the disk.

e) To verify that your expression is plausible, check what it gives at a very large distance from the disk.

Answer: a)



b) $Q = \int \sigma(\vec{r}')\, da'$

Here $\left.\begin{array}{l} 0 \le s' \le R \\ 0 \le \phi' \le 2\pi \end{array}\right\} \quad da' = s'\, ds'\, d\phi'$

Thus $Q = \int_0^R ds' \int_0^{2\pi} d\phi'\, s' \alpha \frac{\cos\phi'}{s'} = \alpha \int_0^R ds' \underbrace{\int_0^{2\pi} \cos\phi'\, d\phi'}_{=0}$

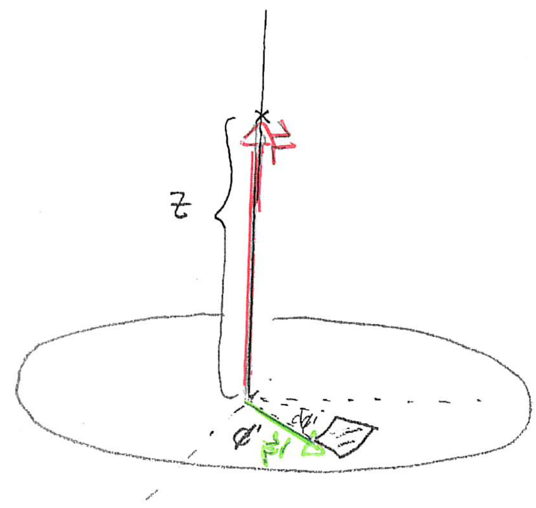
$\Rightarrow \quad Q = 0$

$\boxed{\text{No}}$ $\quad \sigma(\vec{r}')A = \frac{\alpha\cos\phi'}{s'} \pi R^2 \neq 0$

c) Consider the contribution from the segment

$s' \to s' + ds'$

$\phi' \to \phi' + d\phi'$



Then we have to add over all segments

$$\left. \begin{array}{l} 0 \le s' \le R \\ 0 \le \phi' \le 2\pi \end{array} \right\} \quad da' = s' ds' d\phi'$$

The contribution is

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{\mathscr{r}^2} \hat{\mathscr{r}}$$

Charge: $\quad dq = \sigma(\vec{r}')da'$

$$= \alpha \frac{\cos\phi'}{s'} s' ds' d\phi' \quad \Rightarrow \quad dq = \alpha \cos\phi' ds' d\phi'$$

Separation vector $\quad \vec{\mathscr{r}} = \vec{r} - \vec{r}'$ and here (cylindrical)

$$\left. \begin{array}{l} \vec{r} = z\hat{z} \\ \vec{r}' = s'\hat{s} \end{array} \right\} \Rightarrow \vec{\mathscr{r}} = z\hat{z} - s'\hat{s}$$

So $\quad \mathscr{r}^2 = \vec{\mathscr{r}} \cdot \vec{\mathscr{r}} = (z\hat{z} - s'\hat{s}) \cdot (z\hat{z} - s'\hat{s})$

$$= z^2 + s'^2 \quad \Rightarrow \quad \mathscr{r} = (z^2 + s'^2)^{1/2}$$

Thus

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{z\hat{z} - s'\hat{s}}{(z^2 + s'^2)^{1/2}}$$

Thus

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\alpha \cos\phi'}{(z^2+s'^2)^{3/2}} \left(z\hat{z} - s'\hat{s}\right) ds' d\phi'$$

<u>Integrate:</u> Again $\hat{s}$ varies over the surface. Specifically

$$\hat{s} = \cos\phi' \hat{x} + \sin\phi' \hat{y}$$

$$\Rightarrow \quad d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\alpha \cos\phi'}{(z^2+s'^2)^{3/2}} \left[ z\hat{z} - s'\cos\phi' \hat{x} - s'\sin\phi' \hat{y} \right] ds' d\phi'$$

$$\Rightarrow \quad \vec{E} = \frac{\alpha}{4\pi\epsilon_0} \int_0^R ds' \int_0^{2\pi} d\phi' \left\{ \frac{z\cos\phi'}{(z^2+s'^2)^{3/2}} \hat{z} - \frac{s'\cos^2\phi'}{(z^2+s'^2)^{3/2}} \hat{x} - \frac{s'\cos\phi'\sin\phi'}{(z^2+s'^2)^{3/2}} \hat{y} \right\}$$

If we integrate w.r.t $\phi'$ first

$$\int_0^{2\pi} \cos\phi' d\phi' = 0$$

$$\int_0^{2\pi} \cos^2\phi' d\phi' = \pi$$

$$\int_0^{2\pi} \sin\phi' \cos\phi' d\phi' = 0$$

Thus

$$\vec{E} = -\frac{\alpha}{4\pi\epsilon_0} \int_0^R \frac{s'}{(z^2+s'^2)^{3/2}} ds' \hat{x}$$

$$= -\frac{\alpha}{4\pi\epsilon_0} \left(z^2+s'^2\right)^{-1/2} \Big|_0^R \hat{x}$$

So $$\vec{E} = \frac{\alpha}{4\pi\epsilon_0}\left[\frac{1}{(z^2+R^2)^{1/2}} - \frac{1}{(z^2)^{1/2}}\right]\hat{x}$$

$$\Rightarrow \quad \vec{E} = \frac{\alpha}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{|z|}\right]\hat{x}$$

d) Here $z \ll R$ and

$$\vec{E} \approx +\frac{\alpha}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{|z|}\right]\hat{x}$$

$$\approx -\frac{\alpha}{4\pi\epsilon_0}\frac{\hat{x}}{|z|}$$

The direction is consistent

e) Here $z \gg R$ and

$$\frac{1}{(z^2+R^2)^{1/2}} = \frac{1}{\left[z^2\left(1+\frac{R^2}{z^2}\right)\right]^{1/2}} = \frac{1}{|z|}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right)$$

So

$$\vec{E} \approx -\frac{\alpha}{4\pi\epsilon_0}\frac{1}{z}\frac{R^2}{z^2}\hat{x}$$

and the direction is consistent.

## Three dimensional charge distributions

We can extend this framework to charge distributed over a three dimensional region. The charge distribution is described by a volume charge density, $\rho(\vec{r}')$, which is a scalar function with units of $C/m^3$. Again this is defined so that

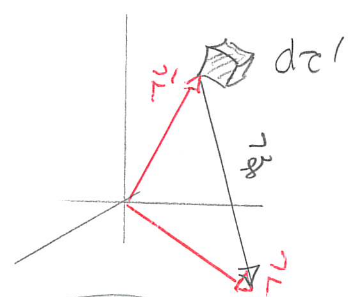
The charge in an infinitesimal region of volume $d\tau'$ located at $\vec{r}'$ is

$$dq = \rho(\vec{r}')d\tau' \qquad \text{as} \qquad d\tau' \to 0$$

More precisely

The charge in an extended region is

$$\text{charge} = \int_{\text{region}} \rho(\vec{r}')\,d\tau'$$




We can then compute the field by

Decompose into segments. Contribution from segment at $\vec{r}'$ to field at $\vec{r}$ is

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

Specify volume element in terms of co-ordinates and give range of co-ordinates over all segments

$$\vec{E} = \int_{\text{all segments}} d\vec{E}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{\text{region}} \frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

Again one can extend the integral to all space. Thus in general in electrostatics:

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\rho(\vec{r}\,')}{\gamma^2} \hat{\gamma} \, d\tau'}$$

Note that:

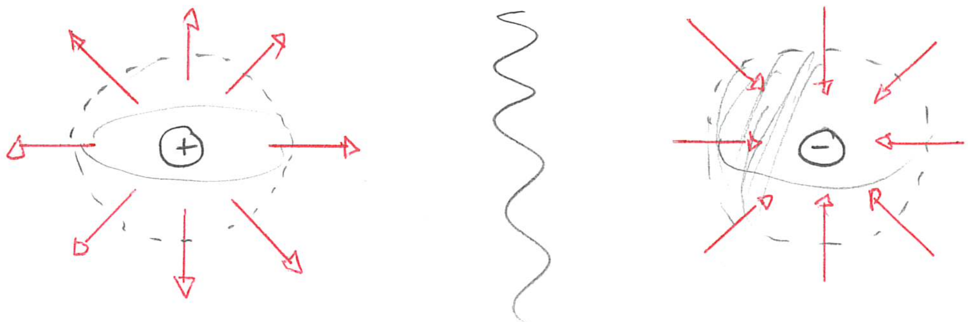
* This ~~DOES NOT~~ mean find a variable $\tau'$ and integrate w.r.t. this single variable

* It DOES mean decompose into pieces and assemble.

* $d\tau'$ refers to three variables. These appear in $\vec{r}\,'$ and thus $\rho(\vec{r}\,')$ and also $\gamma$ (via $\vec{\gamma} = \vec{r} - \vec{r}\,'$).

## Gauss' Law (Introduction)

Consider a single point source charge at the origin. The electric fields are:




We can construct spherical surfaces that enclose the charge. Then the flux of the electric field through these are:

$$\boxed{\text{Flux of } \vec{E} = \oint_{\text{surface}} \vec{E} \cdot d\vec{a}}$$

We see that

$$\oint \vec{E} \cdot d\vec{a} > 0 \quad \text{if source is positive}$$

$$\oint \vec{E} \cdot d\vec{a} < 0 \quad \text{if " " negative}$$

So the flux says something about the enclosed source charge.

**Example:** A point source charge is located at the origin. Determine the flux of the electric field through a sphere of radius $R$ centered at the origin.

**Answer** Here

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

in spherical co-ordinates. For the surface

$$\left.\begin{array}{c} r = R \\ 0 \leq \theta \leq \pi \\ 0 \leq \phi \leq 2\pi \end{array}\right\} \quad \begin{aligned} d\vec{a} &= r^2 \sin\theta \, d\theta \, d\phi \, \hat{r} \\ &= R^2 \sin\theta \, d\theta \, d\phi \, \hat{r} \end{aligned}$$

On the surface $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{r}$. Thus

$$\vec{E}\cdot d\vec{a} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{r}\cdot R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$= \frac{1}{4\pi\epsilon_0} q \sin\theta \, d\theta \, d\phi$$

so

$$\oint \vec{E}\cdot d\vec{a} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, \frac{q}{4\pi\epsilon_0}\sin\theta$$

$$= \frac{q}{4\pi\epsilon_0} \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = q/\epsilon_0$$

Thus we have

$$\oint \vec{E}\cdot d\vec{a} = q/\epsilon_0$$

This is a simple version of what will eventually be Gauss' Law