Fri: HW due

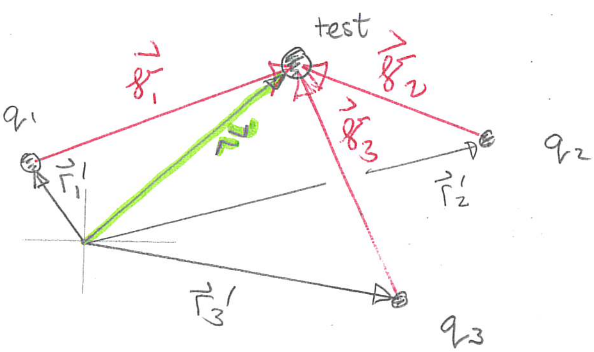
Mon. Read 2.1.4, 2.2.1

Tues: HW 8

Electric fields

The foundation of electrostatics is Coulomb's Law. If there are multiple source charges then the force on a test charge is

$$\vec{F} = \sum_i \frac{1}{4\pi\epsilon_0} \frac{q_i Q}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$



where the separation vectors are

$$\vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i'$$

We can reexpress this as

$$\vec{F} = \underbrace{Q}_{\text{only depends on test}} \underbrace{\sum_i \frac{1}{4\pi\epsilon_0} \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i}_{\text{can be calculated using source charges and any possible test location}}$$

We therefore define

> The electric field produced at location $\vec{r}$ by a collection of point source charges is
>
> $$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$
>
> q_i → charge of source i
>
> where $\vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i'$
>
> $\vec{r}_i'$ → location of source i

Demo: PhET Charges + fields. ~ fields produced by various sources

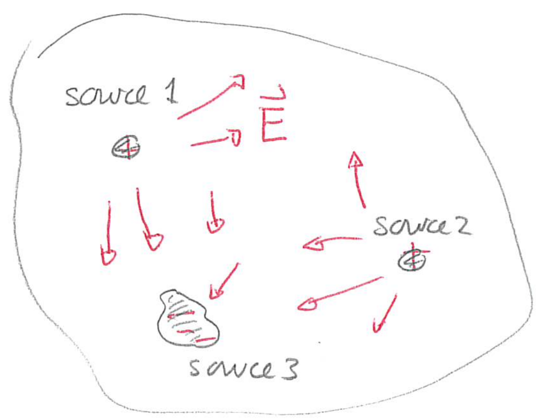
We can then see:

> If a test charge is placed at location $\vec{r}$ then the force exerted by the source electric field on the test charge is:
>
> $$\vec{F} = Q\vec{E}(\vec{r})$$
>
> where $Q =$ charge of test charge
>
> $\vec{E}(\vec{r}) =$ electric field produced by source charges at $\vec{r}$.

This rule was derived for a field produced by an ensemble of point source charges. However, it is true for any arrangement of stationary source charges. The scheme is:

A collection (hidden?) of fixed source charges produces an electric field $\vec{E}(\vec{r})$

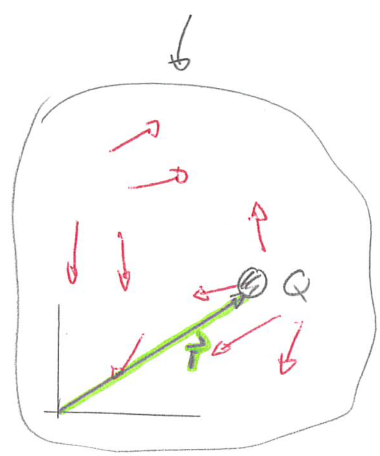
$\Rightarrow$ one vector at each location *vector field*



If a test charge is placed at a particular location, $\vec{r}$, then the source charges (or fields) exert a force

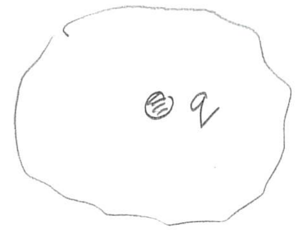
$$\vec{F} = Q\vec{E}(\vec{r})$$

where Q is the charge of the test

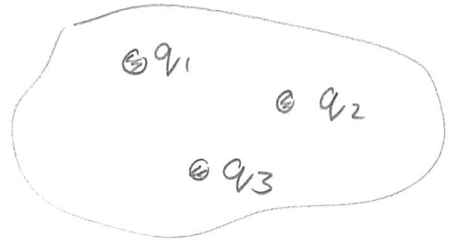


In electrostatics, the challenging and crucial task is to compute the electric fields produced by source charges. Some possible situations are:

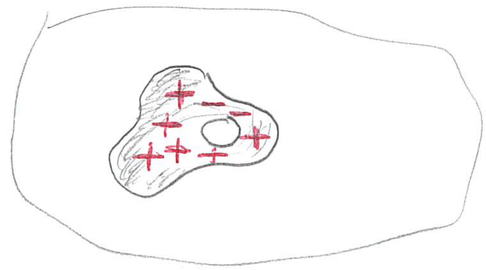
1) source $\equiv$ single point charge



2) source $\equiv$ multiple point charges



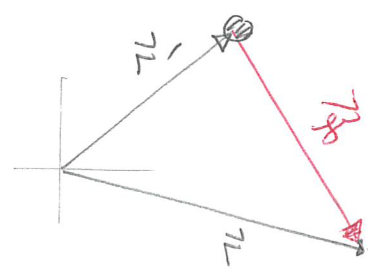
3) source $\equiv$ continuous distribution of charges



Point sources

The derivation provides the basic rule:

> The electric field produced at location $\vec{r}$ by a single point source charge at location $\vec{r}\,'$ is:
>
> $$\vec{E} = \frac{1}{4\pi\epsilon_0}\,\frac{q}{\mathscr{r}^2}\,\hat{\mathscr{r}}$$
>
> where q = the charge of the point source
> $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

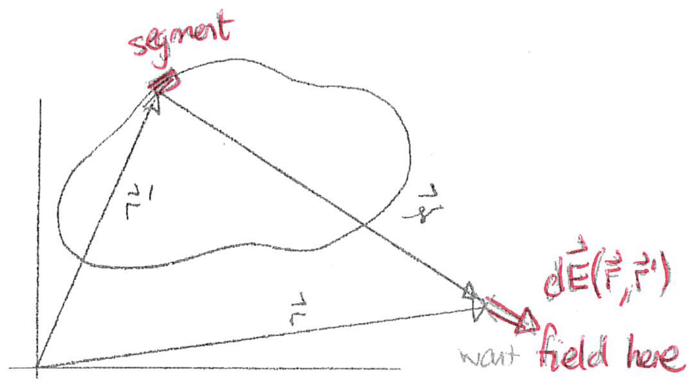


The superposition principle then states that the field produced by multiple point source charges is the vector sum of the fields produced according to the rule above.

Continuous distributions

All physical charge arises from distributions of subatomic point charges. However, at a macroscopic level the density of these is such that we can regard the charge as being continuously and smoothly distributed. We adapt the basic Coulomb's Law point charge rule to compute fields from continuous distributions according to the following scheme.

Consider a one dimensional continuous charge distribution



Break the distribution into infinitesimally small pointlike segments. Each is approximately a point charge

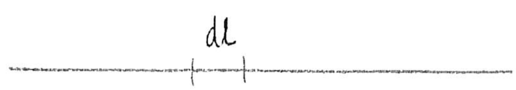
Denote the contribution from the segment at $\vec{r}\,'$ to the field at $\vec{r}$ by

$$d\vec{E}(\vec{r},\vec{r}\,')$$

Add contributions from all segments

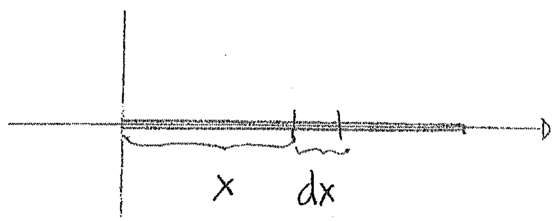
$$\vec{E}(\vec{r}) = \int_{\text{all } \vec{r}\,'} d\vec{E}(\vec{r},\vec{r}\,')$$

To render the calculation tractable, we need a mathematical description of how the charge is distributed. We can do so via a linear charge density λ measured in units of C/m. The simplest example is a uniformly charged straight wire with length L and charge Q. Then $\lambda = Q/L$. Note that the charge in a segment of length dl is



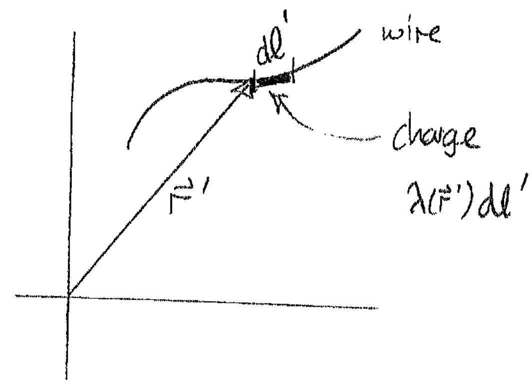
$$\lambda\, dl$$

We generalize this to situations where the charge is not uniformly distributed. Then the charge density depends on the location along the distribution. For a one dimensional wire along x, the charge from $x \to x+dx$ is



$$\lambda(x)\,dx$$

In general the charge density at location $\vec{r}\,'$, denoted $\lambda(\vec{r}\,')$ has meaning



charge in segment of length dl' starting at $\vec{r}\,'$ is $\lambda(\vec{r}\,')\,dl'$

1 Field at the center of a ring of charge

A ring of radius R lies in the xy plane with its center at the origin. The linear charge density on the ring is

$$\lambda(\mathbf{r}') = \frac{Q}{4R} \sin\phi'$$

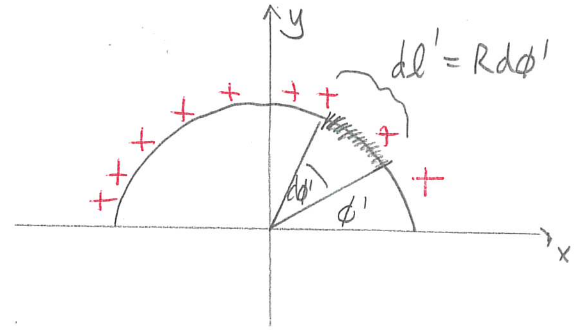
where ϕ' is the angle counterclockwise from the $+x$ axis.

a) Determine the charge in the upper half ($y > 0$) of the ring.

b) Determine the total charge around the entire ring

c) Determine the electric field at the center of the ring.

Answer: a) For the half in the $y > 0$ region use cylindrical co-ords



$$s' = R$$

$$0 \leq \phi' \leq \pi$$

Now consider a segment from $\phi' \to \phi' + d\phi'$. The length of this segment is $dl' = Rd\phi'$. The charge in this segment is

$$dq = \lambda(\vec{r}')\, dl'$$

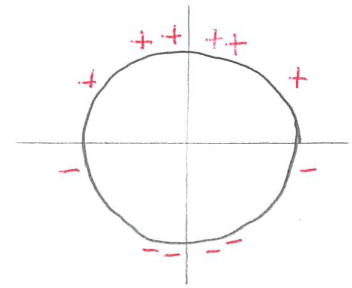
$$= \frac{Q}{4R} \sin\phi' R d\phi' = \frac{Q}{4} \sin\phi' d\phi'$$

Then in the entire half, the charge is:

$$\text{charge} = \int dq = \int_0^{\pi} \frac{Q}{4} \sin\phi' d\phi' = -\frac{Q}{4} \cos\phi' \Big|_0^{\pi} = \frac{Q}{2}$$

$=0$ $\boxed{\text{charge} = Q/2}$

b) For the entire ring the process is similar. Then

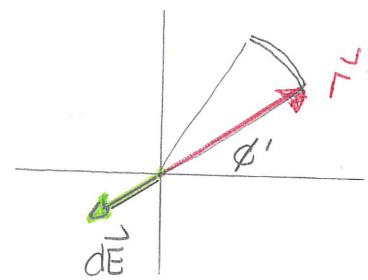


$$s' = R$$
$$0 \leq \phi' \leq 2\pi$$

$$\text{charge} = \int_0^{2\pi} \frac{Q}{4} \sin\phi' \, d\phi' = -\frac{Q}{4} \cos\phi' \Big|_0^{2\pi} = 0 \quad \Rightarrow \quad \text{charge} = 0$$

The symmetry of the situation supports this result.

c) We consider a segment from ϕ' to $\phi' + d\phi'$. Then this is approximately a point charge and produces field contribution



$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{\mathscr{r}^2} \hat{\mathscr{r}}$$

We need $\vec{\mathscr{r}}$ to compute this. Then $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and

$$\left.\begin{array}{l} \vec{r} = 0 \\ \vec{r}\,' = R\hat{s} \end{array}\right\} \Rightarrow \quad \vec{\mathscr{r}} = 0 - R\hat{s} = -R\hat{s}$$

Thus $\mathscr{r} = R$ and $\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = -\hat{s} \Rightarrow \hat{\mathscr{r}} = -\hat{s}$.

So

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{R^2} (-\hat{s})$$

But

$$dq = \lambda(\vec{r}\,')\, dl' = \frac{Q}{4R} \sin\phi' \, R \, d\phi' = \frac{Q}{4} \sin\phi' \, d\phi'$$

Thus the contribution is

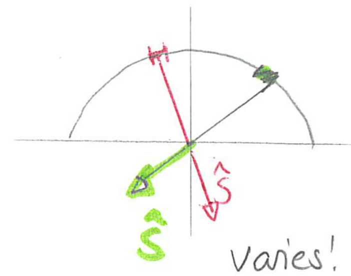
$$d\vec{E} = -\frac{1}{4\pi\epsilon_0} \frac{Q}{4R^2} \sin\phi' \, d\phi' \, \hat{s}$$

Now we must add over all segments:

$$\vec{E} = \int d\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{Q}{4R^2}\int_0^{2\pi} \sin\phi'\, d\phi'\, \hat{s}$$

It may seem that we can extract $\hat{s}$ but this is not constant. Note that

$$\hat{s} = \cos\phi'\,\hat{x} + \sin\phi'\,\hat{y}$$



Thus:

$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{Q}{4R^2}\left[\int_0^{2\pi} \sin\phi'\cos\phi'\, d\phi'\,\hat{x} + \int_0^{2\pi} \sin^2\phi'\, d\phi'\,\hat{y}\right]$$

Now $\displaystyle\int_0^{2\pi} \sin\phi'\cos\phi'\, d\phi' = \frac{1}{2}\sin^2\phi' \Big|_0^{2\pi} = 0$

$\cos 2\theta = 1 - 2\sin^2\theta$

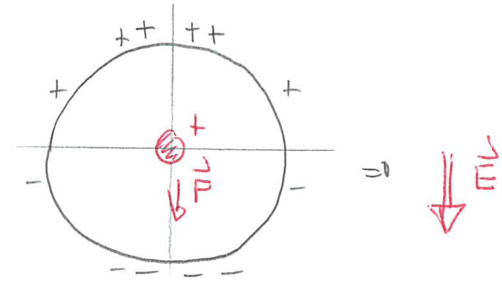
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\int_0^{2\pi} \sin^2\phi'\, d\phi' = \int_0^{2\pi} \frac{1}{2}(1-\cos 2\phi')\, d\phi'$$

$$= \frac{1}{2}\left[\phi' - \frac{1}{2}\sin 2\phi'\right]_0^{2\pi} = \pi.$$

Thus $\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{Q}{4R^2}\pi\,\hat{y} \Rightarrow \vec{E} = -\frac{Q}{16\epsilon_0 R^2}\hat{y}$ $\boxed{}$

Note that the symmetry assures us that the field points along $-\hat{y}$



Notation for continuous charge distribution + fields

In general the total charge is:

$$\boxed{Q = \int \lambda(\vec{r}\,')\, dl'}$$

and the total field is

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\, \hat{\mathscr{r}}\, dl'}$$

These may appear to mean a single variable integration w.r.t a variable l'. This is not true.

These DO NOT mean integrate w.r.t. variable l'

These DO mean do the entire decompostion + all steps.