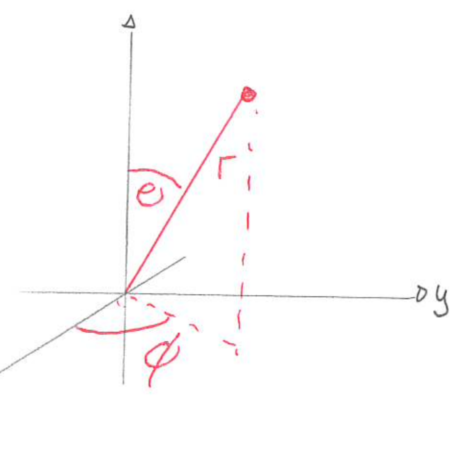
Tues: HW 6

Weds: Read 2.1.1 → 2.1.3

## Spherical Co-ordinates

For sphericaly symmetric situations it is convenient to use spherical co-ordinates defined via:




$r =$ distance from origin to point

$\theta =$ angle from $+\hat{z}$ axis to origin/point line

$\phi =$ angle in xy plane to point

Then precisely

$$\boxed{\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{aligned} \quad \text{with} \quad \begin{aligned} &0 \le r < \infty \\ &0 \le \theta \le \pi \\ &0 \le \phi \le 2\pi \end{aligned}}$$

Inverting these gives:

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \phi &= \arctan\left(y/x\right) \\ \theta &= \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right) \end{aligned}}$$

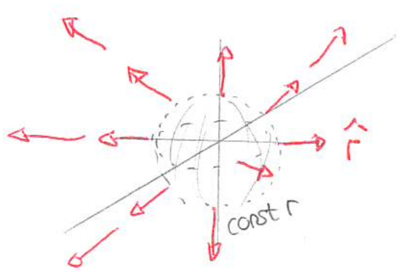
We again construct unit vectors, line elements, areas and volumes. The unit vectors are:

$\hat{r}$ = unit vector perpendicular to constant $r$ in direction of increasing $r$

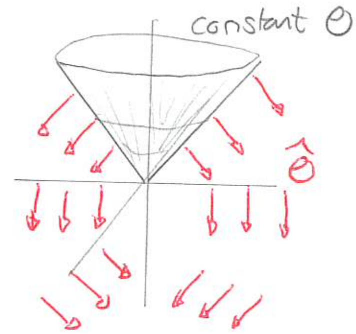
$\hat{\theta}$ = " " " " " $\theta$ " " " " $\theta$

$\hat{\phi}$ = " " " " " $\phi$ " " " " $\phi$

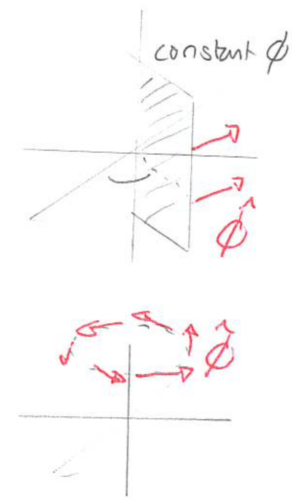
We depict these as:




$\hat{r}$ = radially outward




$\hat{\theta}$ circles in any plane containing z axis




$\hat{\phi}$ = circles about z axis

In order to relate these to Cartesian unit vectors we can use:

1) geometrical construction

2) differential geometry:

$$\hat{r} = \left[\frac{\partial x}{\partial r}\hat{x} + \frac{\partial y}{\partial r}\hat{y} + \frac{\partial z}{\partial r}\hat{z}\right] \Big/ \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}$$

$$\hat{\theta} = \left(\frac{\partial x}{\partial \theta}\hat{x} + \frac{\partial y}{\partial \theta}\hat{y} + \frac{\partial z}{\partial \phi}\hat{z}\right) \Big/ \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2}$$

$$\hat{\phi} = \left(\frac{\partial x}{\partial \phi}\hat{x} + \frac{\partial y}{\partial \phi}\hat{y} + \frac{\partial z}{\partial \phi}\hat{z}\right) \Big/ \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2}$$

Thus:

$$\hat{r} = \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\phi\cos\theta\,\hat{x} + \sin\phi\cos\theta\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

with inverse relationships

$$\hat{x} = \cos\phi\sin\theta\,\hat{r} + \cos\phi\cos\theta\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

$$\hat{y} = \sin\phi\sin\theta\,\hat{r} + \sin\phi\cos\theta\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

These satisfy:

| | | |
|---|---|---|
| $\hat{r}\cdot\hat{r} = 1$ | $\hat{r}\cdot\hat{\theta} = 0$ | $\hat{r}\times\hat{\theta} = \hat{\phi}$ |
| $\hat{\theta}\cdot\hat{\theta} = 1$ | $\hat{r}\cdot\hat{\phi} = 0$ | $\hat{\theta}\times\hat{\phi} = \hat{r}$ |
| $\hat{\phi}\cdot\hat{\phi} = 1$ | $\hat{\theta}\cdot\hat{\phi} = 0$ | $\hat{\phi}\times\hat{r} = \hat{\theta}$ |

## 1 Vectors in spherical coordinates

Let

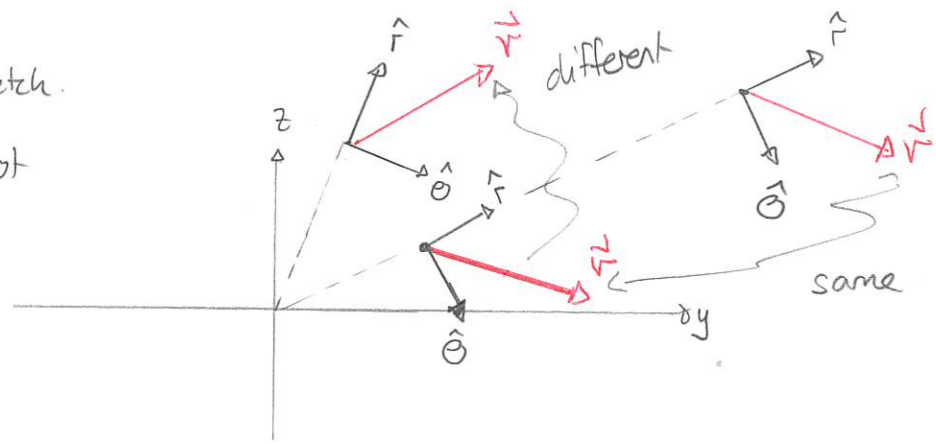
$$\mathbf{v} := 2\hat{\mathbf{r}} + 2\hat{\boldsymbol{\theta}}.$$

a) Is this vector constant? Check by sketching the vector at various locations in the $yz$ plane.

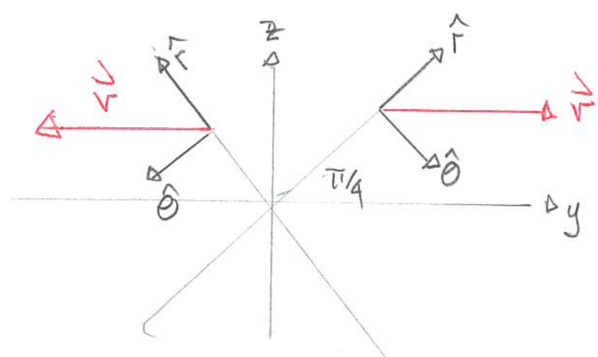
b) Are there any locations such that the vectors $\mathbf{v}$ at the locations are opposite to each other?

Answer: a) sketch.

Shows, not constant




b)




At these points along a 45° cone, yes.

# Integration

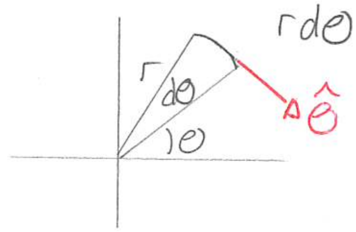
We require line elements to do line integrals. Special cases are:

1) along $\hat{r}$ ($\theta, \phi$ const)

$$\vec{dl} = dr\,\hat{r}$$

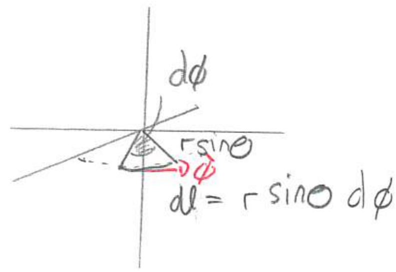
2) along $\hat{\theta}$ ($r, \phi$ constant)

$$\vec{dl} = r\,d\theta\,\hat{\theta}$$




3) along $\hat{\phi}$ ($r, \phi$ const)

$$\vec{dl} = r\sin\theta\,d\phi\,\hat{\phi}$$



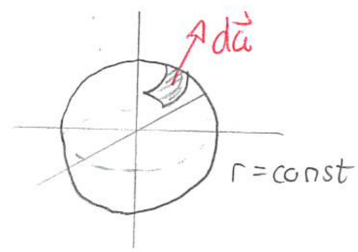

Together:

$$\boxed{\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}}$$

Special area elements are

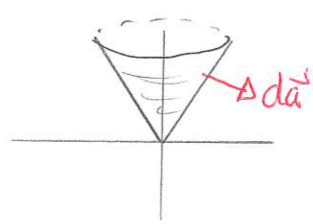
1) $r$ constant (spherical surface)

$$d\vec{a} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$$



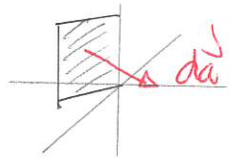

2) $\theta$ constant (conical surface)

$$d\vec{a} = r\sin\theta\,dr\,d\phi\,\hat{\theta}$$




3) $\phi$ constant (plane)

$$d\vec{a} = r\,dr\,d\theta\,\hat{\phi}$$




The volume element is:

$$\boxed{d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi}$$

**2 Divergence theorem in spherical coordinates**

Let

$$\mathbf{v} := \frac{1}{r}\hat{\theta} + \frac{1}{r}\hat{\phi}.$$

Consider the surface with three surfaces:

1. the quarter sphere, centered at the origin with radius $a$,
2. surface in the plane $z = 0$ for which $-\sqrt{a^2 - x^2} \leqslant y \leqslant 0$, and
3. surface in the plane $y = 0$ for which $0 \leqslant z \leqslant \sqrt{a^2 - x^2}$.

a) Determine $\oint_S \mathbf{v} \cdot \mathbf{da}$ over this surface.

b) Verify the divergence theorem for this example.

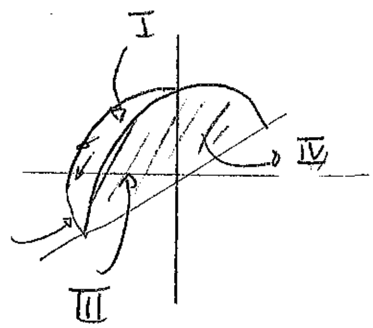
Answer: Surface has base faces

I curve

II base

III face at $y=0$ $x>0$

IV " at $y=0$ $x<0$




Surface I

$$\left.\begin{array}{l} r = a \\ 0 \leq \theta \leq \pi/2 \\ \pi \leq \phi \leq 2\pi \end{array}\right\} \quad d\vec{a} = r^2 \sin\theta \, d\theta d\phi \, \hat{r} = a^2 \sin\theta \, d\theta d\phi \, \hat{r}$$

So $\vec{v} \cdot d\vec{a} = \left(\frac{1}{r}\hat{\theta} + \frac{1}{r}\hat{\phi}\right) \cdot a^2 \sin\theta \, d\theta d\phi \, \hat{r} = 0 \Rightarrow \int_{I} \vec{v} \cdot d\vec{a} = 0$

Surface II

$$\left.\begin{array}{l} 0 \leq r \leq a \\ \theta = \pi/2 \\ \pi \leq \phi \leq 2\pi \end{array}\right\} \quad d\vec{a} = r\underbrace{\sin\theta}_{1} \, dr d\phi \, \hat{\theta} \Rightarrow \vec{v} \cdot d\vec{a} = \frac{r}{r} dr d\phi = dr d\phi$$

$$\int_{II} \vec{v} \cdot d\vec{a} = \int_0^a dr \int_{\pi}^{2\pi} d\phi = \pi a \Rightarrow \int_{II} \vec{v} \cdot d\vec{a} = \pi a$$

Surface III

$$\left.\begin{array}{c} 0 \le r \le a \\ 0 \le \theta \le \pi/2 \\ \phi = 0 \end{array}\right\} \quad d\vec{a} = r\,dr\,d\theta\,\hat{\phi}$$

$$\vec{v}\cdot d\vec{a} = \frac{1}{r}\, r\,dr\,d\theta = dr\,d\theta$$

$$\int_{III} \vec{v}\cdot d\vec{a} = \int_0^a dr \int_0^{\pi/2} d\theta = \frac{\pi}{2}a \quad \Rightarrow \int_{III} \vec{v}\cdot d\vec{a} = \frac{\pi}{2}a$$

Surface IV

$$\left.\begin{array}{c} 0 \le r \le a \\ 0 \le \theta \le \pi/2 \\ \phi = \pi \end{array}\right\} \quad d\vec{a} = -r\,dr\,d\theta\,d\phi$$

$$\vec{v}\cdot d\vec{a} = -\frac{1}{r}\, r\,dr\,d\theta = -dr\,d\theta$$

$$\int_{IV} \vec{v}\cdot d\vec{a} = -\int_0^a dr \int_0^{\pi/2} d\theta = -\frac{\pi}{2}a \quad \Rightarrow \int_{IV} \vec{v}\cdot d\vec{a} = -\pi/2\, a$$

So

$$\oint \vec{v}\cdot d\vec{a} = 0 + \pi a + \frac{\pi}{2}a - \frac{\pi}{2}a \quad \Rightarrow \boxed{\oint \vec{v}\cdot d\vec{a} = \pi a}$$

b) In general

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Here $v_r = 0$

$v_\theta = 1/r$

$v_\phi = 1/r$

Thus

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\cancel{\left(r^2 0\right)}^{0} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin\theta\right) + \frac{1}{r\sin\theta}\cancel{\frac{\partial}{\partial\phi}\left(\frac{1}{r}\right)}^{0}$$

$$= \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}$$

Then for the volume

$$\left.\begin{array}{l} 0 \le r \le a \\ 0 \le \theta \le \pi/2 \\ \pi \le \phi \le 2\pi \end{array}\right\} \quad d\tau = r^2\sin\theta\, dr\, d\theta\, d\phi$$

and

$$\int \vec{\nabla}\cdot\vec{v}\, d\tau = \int_0^a dr \int_0^{\pi/2} d\theta \int_\pi^{2\pi} d\phi\ \cancel{r^2}\cancel{\sin\theta}\ \frac{1}{\cancel{r^2}}\frac{\cos\theta}{\cancel{\sin\theta}}$$

$$= \underbrace{\int_0^a dr}_{a}\ \underbrace{\int_0^{\pi/2}\cos\theta\, d\theta}_{\underbrace{\sin\theta\Big|_0^{\pi/2}}_{1}}\ \underbrace{\int_\pi^{2\pi} d\phi}_{\pi}$$

$$\Rightarrow \int \vec{\nabla}\cdot\vec{v}\, d\tau = \pi a$$

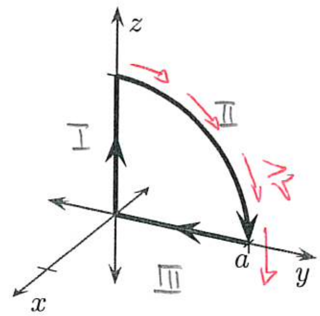
## 3 Line integration in spherical coordinates

Let

$$\mathbf{v} := r^2\hat{\boldsymbol{\theta}}.$$

a) Determine $\oint \mathbf{v}\cdot d\mathbf{l}$ along the curve.

b) Verify Stokes' theorem for this example.




a) Three parts

Do $\int$ then negate

| I | II | III |
|---|---|---|
| $0 \le r \le a$, $\theta = 0$, $\phi = 0$ | $r = a$, $0 \le \theta \le \pi/2$, $\phi = \pi/2$ | $0 \le r \le a$, $\theta = \pi/2$, $\phi = \pi/2$ |
| $\vec{dl} = dr\,\hat{r}$ | $\vec{dl} = r\,d\theta\,\hat{\theta}$ | $\vec{dl} = dr\,\hat{r}$ |
| $\vec{v}\cdot\vec{dl} = 0$ | $\vec{v}\cdot\vec{dl} = r^3 d\theta = a^3 d\theta$ | $\vec{v}\cdot\vec{dl} = 0$ |
| $\int_{I}\vec{v}\cdot\vec{dl} = 0$ | $\int_{II}\vec{v}\cdot\vec{dl} = \int_0^{\pi/2} a^3 d\theta = \frac{\pi a^3}{2}$ | $\int_{III}\vec{v}\cdot\vec{dl} = 0$ |

So $\oint \vec{v}\cdot\vec{dl} = \int_{I}\vec{v}\cdot\vec{dl} + \int_{II}\vec{v}\cdot\vec{dl} + \int_{III}\vec{v}\cdot\vec{dl} \Rightarrow \oint \vec{v}\cdot\vec{dl} = \frac{\pi a^3}{2}$

b) For spherical

$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

$V_r = 0$

$V_\theta = r^2$

$V_\phi = 0$

gives

$$\vec{\nabla} \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\cancel{\sin\theta\, 0}^{\,0}) - \cancel{\frac{\partial r^2}{\partial\phi}}^{\,0}\right]\hat{r} + \frac{1}{r}\left[\cancel{\frac{1}{\sin\theta}\frac{\partial 0}{\partial\phi}}^{\,0} - \cancel{\frac{\partial}{\partial r}(r0)}^{\,0}\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\, r^2) - \cancel{\frac{\partial}{\partial\theta}0}^{\,0}\right]\hat{\phi} \quad = \quad \frac{1}{r}\frac{\partial}{\partial r} r^3\,\hat{\phi}$$

$$= \quad 3r\,\hat{\phi}$$

Then for the area

$$\left.\begin{array}{l} 0 \le r \le a \\ 0 \le \theta \le \pi/2 \\ \phi = \pi/2 \end{array}\right\} \quad d\vec{a} = \quad r\,dr\,d\theta \quad d\phi$$

So $\quad \vec{\nabla} \times \vec{v} \cdot d\vec{a} = 3r^2 dr\,d\theta$

$$\int \vec{\nabla} \times \vec{v} \cdot d\vec{a} = \int_0^a dr \int_0^{\pi/2} d\theta\; 3r^2$$

$$= \underbrace{\int_0^a 3r^2\,dr}_{a^3} \underbrace{\int_0^{\pi/2} d\theta}_{\pi/2}$$

$$\Rightarrow \quad \int \vec{\nabla} \times \vec{v} \cdot d\vec{a} = \frac{\pi}{2} a^3$$