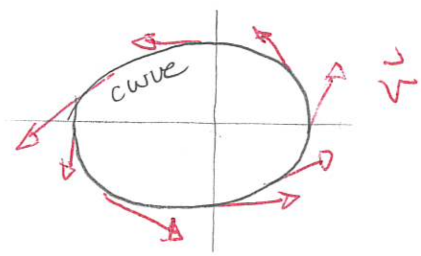
Mon: Read 1.4.1

Tues: HW by 5pm

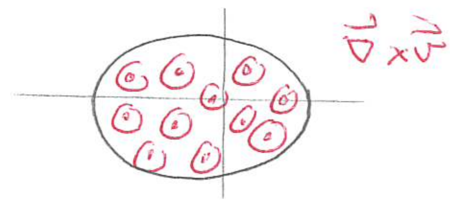
Stokes' Theorem

We now consider line integrals of fields the circle. These cannot be gradients of a scalar and the fundamental theorem for gradients will not apply to evaluate

$$\oint \vec{v} \cdot d\vec{l}$$



We might expect that such integrals are related to the curl of $\vec{v}$. We know that typically



$$\oint_{\text{loop}} \vec{v} \cdot dl \neq 0 \qquad \text{and} \qquad \int_{\text{surface}} \vec{\nabla} \times \vec{v} \cdot d\vec{a} \neq 0$$

Stokes' theorem relates these. It uses the convention for area vector direction in relation to loop direction and gives

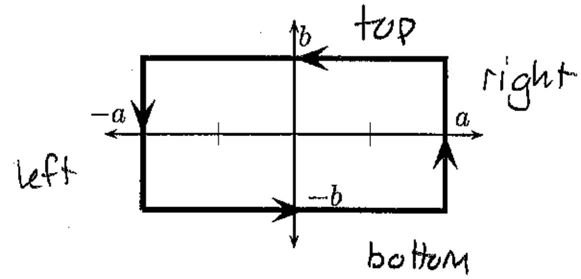
area vector convention.

> Let S be any surface with a boundary loop. Then:
>
> $$\int_{\text{surface}} \vec{\nabla} \times \vec{v} \cdot d\vec{a} = \oint_{\text{boundary loop.}} \vec{v} \cdot d\vec{l}$$

1 Stokes' theorem

Let

$$\mathbf{v} = -\frac{y}{2}\,\hat{\mathbf{x}} + \frac{x}{2}\,\hat{\mathbf{y}}.$$



a) Determine $\oint \mathbf{v}\cdot d\mathbf{l}$ around the loop.

b) Determine $\int \nabla\times\mathbf{v}\cdot d\mathbf{a}$ for the flat surface enclosed by the loop. Verify Stoke's theorem.

Answer: $$\oint_{\text{loop}} \vec{v}\cdot d\vec{l} = \int_{\text{right}} \vec{v}\cdot d\vec{l} + \int_{\text{top}} \vec{v}\cdot d\vec{l} + \int_{\text{left}} \vec{v}\cdot d\vec{l} + \int_{\text{bottom}} \vec{v}\cdot d\vec{l}$$

Right edge

$x = a$

$-b \le y \le b$

$z = 0$

$$d\vec{l} = dy\,\hat{y}$$

$$\vec{v} = -\frac{y}{2}\hat{x} + \frac{x}{2}\hat{y} = -\frac{y}{2}\hat{x} + \frac{a}{2}\hat{y}$$

$$\vec{v}\cdot d\vec{l} = \left(-\frac{y}{2}\hat{x} + \frac{a}{2}\hat{y}\right)\cdot dy\,\hat{y}$$

$$= \frac{a}{2}dy$$

$$\Rightarrow \int \vec{v}\cdot d\vec{l} = \frac{a}{2}\int_{-b}^{b} dy = ab$$

Top edge

$$\int_{\leftarrow} \vec{v}\cdot d\vec{l} = -\int_{\rightarrow} \vec{v}\cdot d\vec{l}$$

evaluate this

$-a \le x \le a$

$y = b$

$z = 0$

$$d\vec{l} = dx\,\hat{x}$$

$$\vec{v} = -\frac{y}{2}\hat{x} + \frac{x}{2}\hat{y} = -\frac{b}{2}\hat{x} + \frac{x}{2}\hat{y}$$

$$\vec{v}\cdot d\vec{l} = -\frac{b}{2}dx$$

$$\int_{\rightarrow} \vec{v}\cdot d\vec{l} = \int_{-a}^{a}\left(-\frac{b}{2}\right)dx = -\frac{b}{2}2a = -ab$$

$$\Rightarrow \int_{\leftarrow} \vec{v}\cdot d\vec{l} = ab$$

Left edge

$$\int_{\downarrow} \vec{v}\cdot d\vec{l} = -\int_{\uparrow} \vec{v}\cdot d\vec{l}$$

evaluate

$x = a$

$-b \le y \le b \qquad d\vec{l} = dy\,\hat{y}$

$z = 0$

$$\vec{v} = -\frac{y}{2}\hat{x} + \frac{a}{2}\hat{y}$$

$$\vec{v}\cdot d\vec{l} = -\frac{a}{2}dy$$

$$\int_{\uparrow} \vec{v}\cdot d\vec{l} = -\int_{-b}^{b} \frac{a}{2}\,dy = -\frac{a}{2}2b = ab$$

$$\Rightarrow \int_{\downarrow} \vec{v}\cdot d\vec{l} = ab$$

Thus $\oint \vec{v}\cdot d\vec{l} = \boxed{4ab}$

Bottom edge

$-a \le x \le a$

$y = -b \qquad d\vec{l} = dx\,\hat{x}$

$z = 0$

$$\vec{v} = +\frac{b}{2}\hat{x} - \frac{x}{2}\hat{y}$$

$$\vec{v}\cdot d\vec{l} = \frac{b}{2}dx$$

$$\int \vec{v}\cdot d\vec{l} = \int_{-a}^{a} \frac{b}{2}dx = ab$$

b)
$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{2} & \frac{x}{2} & 0 \end{vmatrix} = \hat{z}$$

surface $\underbrace{\begin{matrix} -a \le x \le a \\ -b \le y \le b \\ z = 0 \end{matrix}}_{d\vec{a} = dx\,dy\,\hat{z}}$

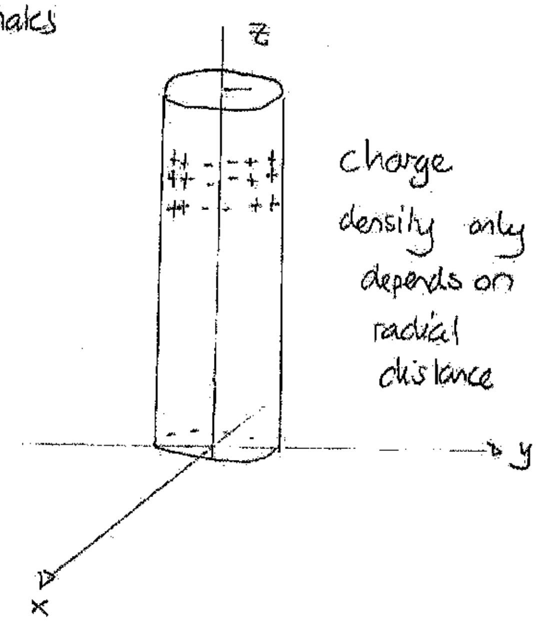
$$\vec{\nabla}\times\vec{v}\cdot d\vec{a} = dx\,dy$$

$$\int \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \int_{-a}^{a} dx \int_{-b}^{b} dy = \boxed{4ab}$$

They match.

Cylindrical co-ordinates

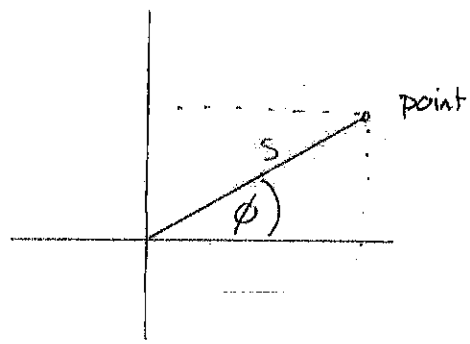
We will encounter situations in which charge and current distributions are symmetrical in a circular fashion under rotations about one axis. In these situations it is less convenient to use Cartesian co-ordinates than a system of co-ordinates adapted to the cylindrical symmetry.



Consider the projection onto the xy plane. We can use polar (or cylindrical) co-ordinates to describe the location of this point.

These are defined implicitly via:

$$\boxed{\begin{aligned} x &= s\cdot\cos\phi \\ y &= s\sin\phi \\ z &= z \\ \text{with } & 0 \le s < \infty \\ & 0 \le \phi < 2\pi \end{aligned}} \approx \boxed{\begin{aligned} s &= \sqrt{x^2+y^2} \\ \phi &= \arctan\left(\tfrac{y}{x}\right) \\ z &= z \end{aligned}}$$

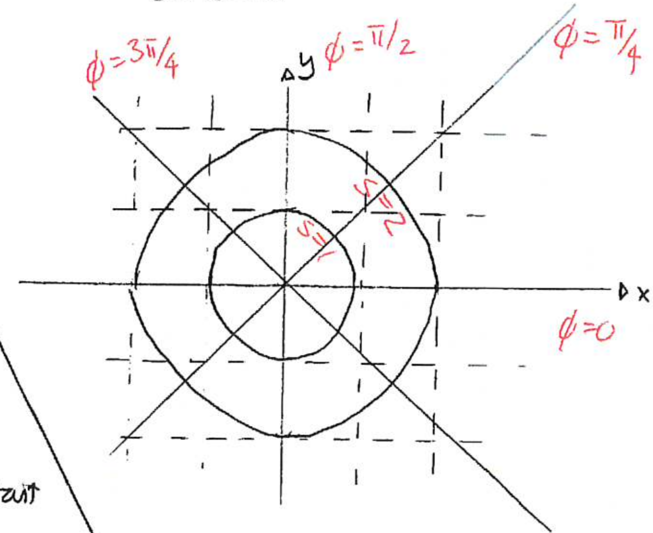


These transformations plus various definitions give rise to:

1) special co-ordinate systems and unit vectors for cylindrical co-ordinates
2) cylindrical versions of line elements $d\vec{l}$
3) cylindrical volume elements
4) cylindrical versions of gradients, divergences + curls

We can establish a new co-ordinate grid that consists of lines along which s, ϕ and z are constant

The unit vectors have definitions:

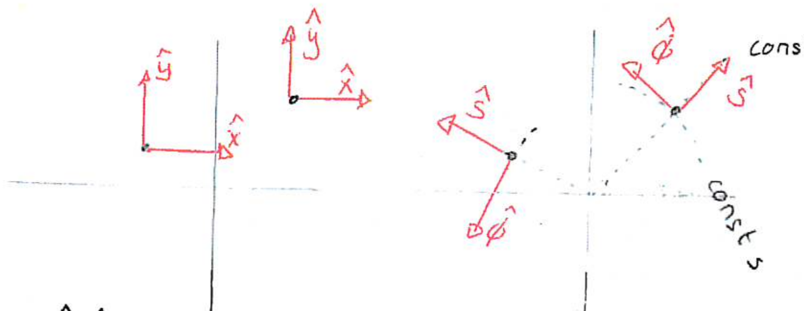


$\hat{s}$ = unit vector tangent to lines of constant ϕ in direction of increasing s

$\hat{\phi}$ = unit vector tangent to lines of constant s in direction of increasing ϕ

Note that these unit vectors vary from one location to another unlike Cartesian vectors $\hat{x}, \hat{y}$.

This will have important implications for divergences, curls, and other derivatives



$\hat{x}, \hat{y}$ same at different locations.

$\hat{s}$ differs from one location to another so does $\hat{\phi}$

The relationships between the cylindrical and Cartesian basis vectors can be determined via geometry + trigonometry:

$$\boxed{\begin{aligned} \hat{s} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \\ \hat{z} &= \hat{z} \end{aligned}} \quad \leftrightarrow \quad \boxed{\begin{aligned} \hat{x} &= \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi} \\ \hat{y} &= \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi} \\ \hat{z} &= \hat{z} \end{aligned}}$$

The basis vectors satisfy:

$$\hat{s}\cdot\hat{s} = 1 \qquad \hat{\phi}\cdot\hat{\phi} = 1 \qquad \hat{s}\times\hat{\phi} = \hat{z}$$

$$\hat{s}\cdot\hat{\phi} = 0 \qquad \hat{z}\cdot\hat{z} = 1 \qquad \hat{\phi}\times\hat{z} = \hat{s}$$

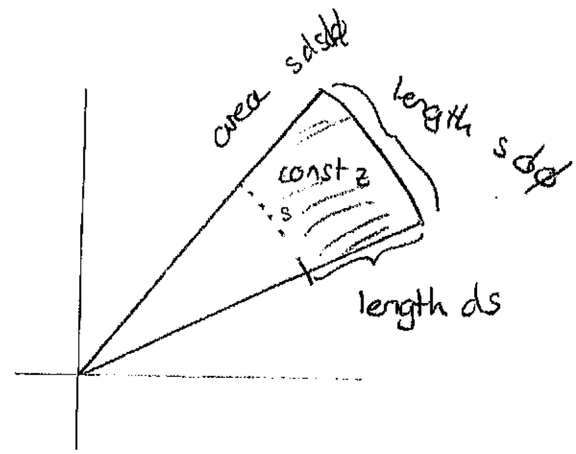
$$\hat{s}\cdot\hat{z} = 0 \qquad \hat{\phi}\cdot\hat{z} = 0 \qquad \hat{z}\times\hat{s} = \hat{\phi}$$

Then we can express any vector as

$$\vec{V} = V_s(s,\phi,z)\,\hat{s} + V_\phi(s,\phi,z)\,\hat{\phi} + V_z(s,\phi,z)\,\hat{z}$$

We can now consider calculus on such vectors. We need expressions for line and volume elements as well as the various three dimensional derivatives. We can get some intuition via two dimensional geometry. We can also derive the results from the basic transformation rules.

Specifically:

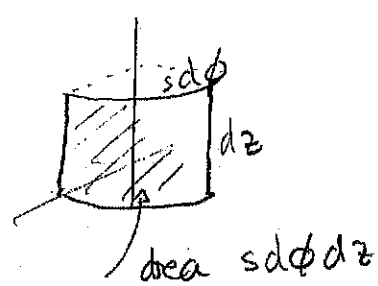


Line element:

$$\boxed{\vec{dl} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}}$$

Surface elements (for specific surfaces)

$z = \text{const}$	$\vec{da} = \pm s\,ds\,d\phi\,\hat{z}$
$\phi = \text{const}$	$\vec{da} = ds\,dz\,\hat{\phi}$
$s = \text{constant}$	$\vec{da} = s\,d\phi\,dz\,\hat{s}$



Volume element

$$\boxed{d\tau = s\,ds\,d\phi\,dz}$$

Proof. (line element)

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$dx = \frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial \phi}d\phi + \cancel{\frac{\partial x}{\partial z}dz} \qquad dy = \frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial \phi}d\phi + \cancel{\frac{\partial y}{\partial z}dz}$$

$$= \cos\phi\,ds - s\sin\phi\,d\phi \qquad\qquad = \sin\phi\,ds + s\cos\phi\,d\phi$$

So

$$dx\,\hat{x} + dy\,\hat{y} = (\cos\phi\,ds - s\sin\phi\,d\phi)(\cos\phi\,\hat{s} - \sin\phi\,\hat{\phi})$$

$$+ (\sin\phi\,ds + s\cos\phi\,d\phi)(\sin\phi\,\hat{s} + \cos\phi\,\hat{\phi})$$

$$= ds\,\hat{s} + s\,d\phi\,\hat{\phi} \qquad \square$$

The others follow in similar fashion.

Gradient, divergence + curl

It may appear that we can adapt gradient divergence + curl via:

$$\vec{\nabla} g = \frac{\partial g}{\partial s}\hat{s} + \frac{\partial g}{\partial \phi}\hat{\phi} + \frac{\partial g}{\partial z}\hat{z}$$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_s}{\partial s} + \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ v_s & v_\phi & v_z \end{vmatrix}$$

But since the basis vectors are position dependent they must be included in the differentiation.

By chaining various calculus rules together, we can reach

$$\vec{\nabla} g = \frac{\partial g}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial g}{\partial \phi}\hat{\phi} + \frac{\partial g}{\partial z}\hat{z}$$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}\left(s v_s\right) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

etc, ...

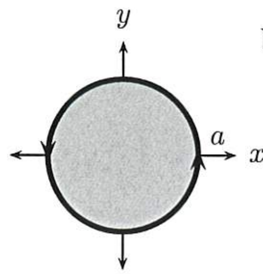
2 Stoke's theorem: cylindrical coordinates

Let

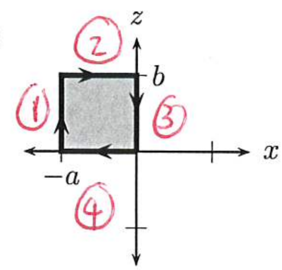
$$\mathbf{v} = s\hat{\boldsymbol{\phi}} + s\cos\phi\,\hat{\mathbf{s}}.$$

Verify Stoke's theorem for the illustrated loops and surfaces.

a)



b)



In both cases we need

$$\vec{\nabla}\times\vec{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{z}$$

and

$$v_\phi = s \qquad v_z = 0$$

$$v_s = s\cos\phi$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{v} = \left[\overset{0}{\cancel{\tfrac{1}{s}0}} - \overset{0}{\cancel{\frac{\partial s}{\partial z}}}\right]\hat{s} + \left[\overset{0}{\cancel{\frac{\partial}{\partial z}s\cos\phi}} - \overset{0}{\cancel{\frac{\partial}{\partial s}0}}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}s^2 - \frac{\partial}{\partial\phi}s\cos\phi\right]\hat{z}$$

$$= \frac{1}{s}\left(2s + s\sin\phi\right)\hat{z}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{v} = \boxed{2+\sin\phi}\,\hat{z}$$

a) line integral

$$\left.\begin{array}{l} s = a \\ 0 \le \phi \le 2\pi \\ z = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \vec{dl} = s\,d\phi\,\hat{\phi} \\ \vec{dl} = a\,d\phi\,\hat{\phi} \end{array}\right\} \Rightarrow \vec{v}\cdot\vec{dl} = sa\,d\phi = a^2\,d\phi$$

$$\Rightarrow \oint \vec{v}\cdot\vec{dl} = \int_0^{2\pi} a^2\,d\phi = \boxed{2\pi a^2}$$

Surface: $\left.\begin{array}{l} 0 \le s \le a \\ 0 \le \phi \le 2\pi \\ z = 0 \end{array}\right\} \Rightarrow d\vec{a} = s\, ds\, d\phi\, \hat{z}$

Thus $\quad \vec{\nabla}\times\vec{v}\cdot d\vec{a} = [2+\sin\phi]\, s\, ds\, d\phi$

$$\int \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \int_0^a ds \int_0^{2\pi} d\phi\; s[2+\sin\phi]$$

$$= \underbrace{\int_0^a s\,ds}_{\frac{a^2}{2}} \; \underbrace{\int_0^{2\pi} [2+\sin\phi]\, d\phi}_{\underbrace{(2\phi-\cos\phi)\big|_0^{2\pi}}_{4\pi}} = 2\pi a^2$$

So $\quad \int \vec{\nabla}\times\vec{v}\cdot d\vec{a} = \oint \vec{v}\cdot d\vec{l}$

b) Here $\phi = \pi$. There are four lines

line	co-ords.	line elem:	$\vec{v}\cdot d\vec{l}$	$\int \vec{v}\cdot d\vec{l}$
1	$s = a$ $0 \le z \le b$	$d\vec{l} = dz\,\hat{z}$	0	0
2	$z = b$ $0 \le s \le a$	$d\vec{l} = ds\,\hat{s}$ outward	$s\cos\phi\, ds$ $= -s\,ds$	$\int_{out} \vec{v}\cdot d\vec{l} = -\int_0^a s\,ds = -a^2/2$ $\int_{in} \vec{v}\cdot d\vec{l} = a^2/2$
3	$s = 0$ $0 < z \le b$	$d\vec{l} = dz\,\hat{z}$ up	0	0
4	$z = 0$ $0 < s \le a$	$d\vec{l} = ds\,\hat{s}$	$s\cos\phi\, ds$ $= -s\,ds$	$\int \vec{v}\cdot d\vec{l} = -a^2/2$

Thus $\oint \vec{v} \cdot d\vec{l} = 0$

Surface

$0 \le s \le a$

$0 \le z \le b$

$\phi = \pi$

$d\vec{a} = -ds\,dz\,\hat{\phi}$

Then $\vec{\nabla} \times \vec{v} \cdot d\vec{a} = 0 \Rightarrow \int \vec{\nabla} \times \vec{v} \cdot d\vec{a} = 0$

Thus

$$\oint \vec{v} \cdot d\vec{l} = \int \vec{\nabla} \times \vec{v} \cdot d\vec{a}$$