Phys 311 2020

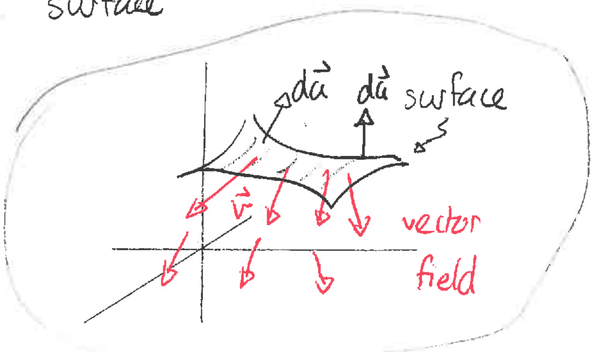
Lecture 8

Fri: HW due by 5pm

Read 1.4.2

Surface Integrals

A surface integral quantifies the flux of a vector field through a surface

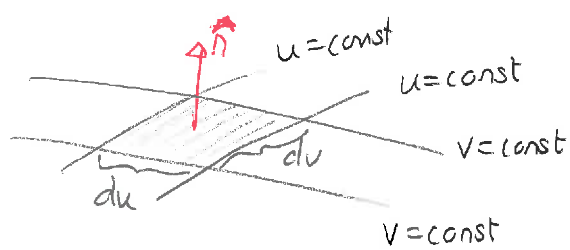


surface integral. Compute $\vec{v}\cdot d\vec{a}$ for each infinitesimal segment and add

$$\int_{\text{surface}} \vec{v}\cdot d\vec{a} = \sum \vec{v}\cdot d\vec{a}$$

The construction requires an area vector $d\vec{a}$ which is perpendicular to the local patch on the surface. This has the following steps

1) parameterize the surface using u, v
2) $da = du\,dv$
3) Construct normal vector $\hat{n}$ (unit vector)
4) $d\vec{a} = du\,dv\,\hat{n}$



We then evaluate $\vec{v}\cdot d\vec{a}$ and after this perform a double integral

$$\int \vec{v}\cdot d\vec{a} = \int\int du\,dv\,(\vec{v}\cdot\hat{n})\ldots$$

Had we been able to parameterize thee surface using x,y then the surface would be described via:

$$z = g(x,y)$$

and eventually:

$$d\vec{a} = \left[-\frac{\partial g}{\partial x}\hat{x} - \frac{\partial g}{\partial y}\hat{y} + \hat{z} \right] dx\, dy$$

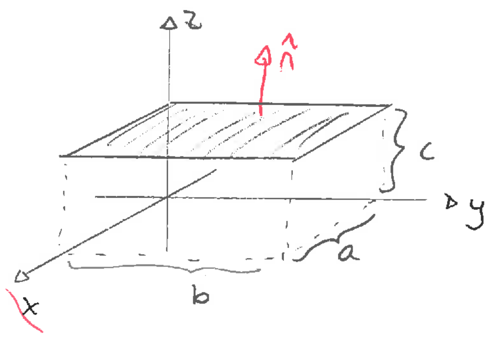
1 Surface integrals

a) Let $\mathbf{v} = z^2 x\hat{\mathbf{x}} + x^2 z\hat{\mathbf{z}}$. Determine the surface integral of $\mathbf{v}$ over the surface $0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b$ and $z = c$.

b) Let $\mathbf{v} = yx^2\hat{\mathbf{x}} + xy^2\hat{\mathbf{y}}$. Determine the surface integral of $\mathbf{v}$ over the flat surface in the region $0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b$ that slants from $z = c$ (at $y = 0$) to $z = 0$ (at $y = b$).

Answer: a) First note variables + surface

$$0 \le x \le a$$
$$0 \le y \le b$$
$$z = c$$



The surface can be parameterized by x, y. Then

$$d\vec{a} = \underbrace{dx\,dy}_{\text{not constant}}\ \hat{z}$$ const variable

Thus

$$\vec{v}\cdot d\vec{a} = (z^2 x\,\hat{x} + z x^2 \hat{z})\cdot dx\,dy\,\hat{z}$$
$$= x^2 z\,dx\,dy$$

But $z = c$

$$\Rightarrow \vec{v}\cdot d\vec{a} = x^2 c\,dx\,dy$$

Then

from

$$\int \vec{v}\cdot d\vec{a} = \int_0^a dx \int_0^b dy\ x^2 c = c\underbrace{\int_0^a x^2 dx}_{a^3/3}\ \underbrace{\int_0^b dy}_{b}$$

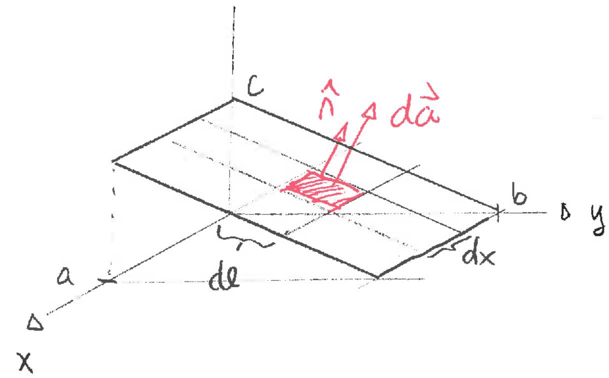
$$\Rightarrow \int \vec{v}\cdot d\vec{a} = \frac{ca^3 b}{3}$$

b) $0 \le x \le a$

$0 \le y \le b$

Along surface

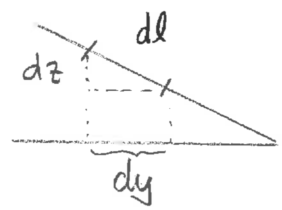
$$z = -\frac{c}{b}x + c$$



We can parameterize the surface using x, y. But the area is

$$d\vec{a} = dx\, dl\, \hat{n}$$

where dl and $\hat{n}$ are as indicated. To get these consider a side view.

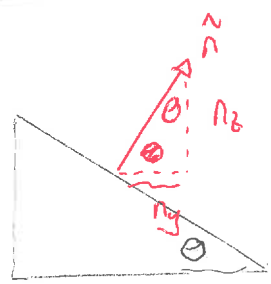


$$dl^2 = dy^2 + dz^2$$

$$= dy^2\left[1 + \left(\frac{dz}{dy}\right)^2\right]$$

$$= dy^2\left[1 + \left(-\frac{c}{b}\right)^2\right] \Rightarrow \boxed{dl = \sqrt{1 + c^2/b^2}\; dy}$$

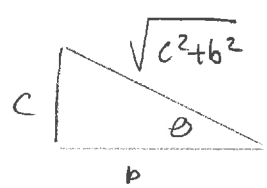
We now need $\hat{n}$



$$\hat{n} = n_y\hat{y} + n_z\hat{z}$$

and

$$\left.\begin{aligned} n_y &= n\sin\theta = \sin\theta \\ n_z &= n\cos\theta = \cos\theta \end{aligned}\right\} \text{ since } n = 1$$



But

$$\cos\theta = \frac{b}{\sqrt{c^2+b^2}} = \frac{b}{b\sqrt{1+c^2/b^2}} = \frac{1}{\sqrt{1+c^2/b^2}}$$

$$\sin\theta = \frac{c}{\sqrt{c^2+b^2}} = \frac{c}{b\sqrt{1+c^2/b^2}} = \frac{c}{b}\,\frac{1}{\sqrt{1+c^2/b^2}}$$

Thus

$$\hat{n} = \frac{1}{\sqrt{1+c^2/b^2}}\left[\frac{c}{b}\hat{y} + \hat{z}\right]$$

$$\Rightarrow \quad d\vec{a} = dx\,dy\sqrt{1+c^2/b^2}\;\frac{1}{\sqrt{1+c^2/b^2}}\left[\frac{c}{b}\hat{y} + \hat{z}\right]$$

$$\Rightarrow \quad d\vec{a} = \left[\frac{c}{b}\hat{y} + \hat{z}\right]dx\,dy$$

Then $\quad \vec{v}\cdot d\vec{a} = \left[yx^2\hat{x} + xy^2\hat{y}\right]\cdot\left[\frac{c}{b}\hat{y} + \hat{z}\right]dx\,dy$

$$= \frac{c}{b}xy^2\,dx\,dy$$

and

$$\int \vec{v}\cdot d\vec{a} = \int_0^a dx\int_0^b dy\;\frac{c}{b}xy^2$$

$$= \frac{c}{b}\int_0^a x\,dx\int_0^b y^2\,dy$$

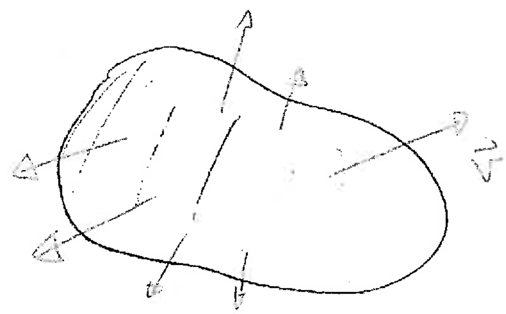
$$= \frac{c}{b}\,\frac{a^2}{2}\,\frac{b^3}{3} \quad = \quad \frac{a^2b^2c}{6}$$

Alternative: with $\quad z = g(x,y) = -\frac{c}{b}y + c$

$$d\vec{a} = \left[-\overset{0}{\cancel{\frac{\partial g}{\partial x}}}\hat{x} - \underbrace{\left(\frac{\partial g}{\partial y}\right)}_{-c/b}\hat{y} + \hat{z}\right]dx\,dy$$

$$= \left[\frac{c}{b}\hat{y} + \hat{z}\right]dx\,dy \ldots$$

Divergence theorem



Consider a surface integral over a closed surface. If $\vec{v}$ diverges from a source within the surface then $\vec{v}\cdot d\vec{a} \geq 0$ along the surface and

$$\oint_{\text{surface}} \vec{v}\cdot d\vec{a} \geq 0$$

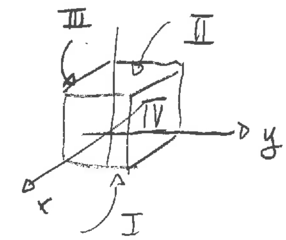
But we also have $\vec{\nabla}\cdot\vec{v} \geq 0$ and we might expect that these are related. In general the divergence theorem states

> For any closed surface S, which bounds a region $\mathcal{V}$
>
> $$\oint_{\text{surface } S} \vec{v}\cdot d\vec{a} = \int_{\text{region } \mathcal{V}} \vec{\nabla}\cdot\vec{v}\, d\tau$$

2 Divergence theorem

Let $\mathbf{v} = x^2y^2\hat{\mathbf{y}}$ and verify the divergence theorem over the cube that bounds $0 \leqslant x, y, z, \leqslant a$.

Answer. There are six surfaces



surface integral

	x	y	z	$d\vec{a}$	$\vec{v}\cdot d\vec{a}$	
I	$0 \to a$	$0 \to a$	0	$-dx\,dy\,\hat{z}$	0	
II	$0 \to a$	$0 \to a$	a	$dx\,dy\,\hat{z}$	0	
III	$0 \to a$	0	$0 \to a$	$-dx\,dz\,\hat{y}$	$-x^2 \underbrace{\cancel{y^2}}_{0} dx\,dz$	$= 0$
IV	$0 \to a$	a	$0 \to a$	$dx\,dz\,\hat{y}$	$+x^2 \underbrace{y^2}_{\to a} dx\,dz$	$= x^2a^2\,dx\,dz$
V	0	$0 \to a$	$0 \to a$	$-dy\,dz\,\hat{x}$	0	
VI	a	$0 \to a$	$0 \to a$	$+dy\,dz\,\hat{x}$	0	

So $$\oint_{\text{all}} \vec{v}\cdot d\vec{a} = \int_{\text{IV}} \vec{v}\cdot d\vec{a} = \int_0^a dx \int_0^a dz\; x^2a^2$$

$$= a^2 \int_0^a x^2 dx \int_0^a dz = \frac{a^6}{3}$$

volume integral

$$\vec{\nabla}\cdot\vec{v} = \cancel{\frac{\partial v_x}{\partial x}} + \frac{\partial v_y}{\partial y} + \cancel{\frac{\partial v_z}{\partial z}} = \frac{\partial}{\partial y} x^2y^2 = 2x^2y$$

Thus $$\int_{\text{region}} \vec{\nabla}\cdot\vec{v}\, d\tau = \int_0^a dx \int_0^a dy \int_0^a dz\; 2x^2y = 2\underbrace{\int_0^a x^2 dx}_{a^3/3} \underbrace{\int_0^a y\,dy}_{a^2/2} \underbrace{\int_0^a dz}_{a}$$

$$= a^6/3$$

MATCH!