Tues: HW 4 due

Weds: Read 1.3.1 (Surface), 1.3.4

Fri: HW 5

Integration theorems for line integrals

Computing ordinary integrals is facilitate by the fundamental theorem of calculus which connects integration to differentiation. Specifically it states that

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

Thus if we want to evaluate

$$\int_a^b g(x)\,dx$$

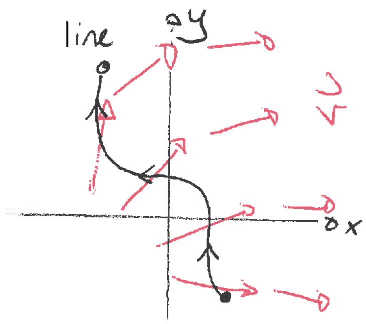
we just need to find $f(x)$ such that $g = \frac{df}{dx}$. Such a function is called the antiderivative of $f(x)$. If this can be determined then the integral would yield $f(b) - f(a)$

We aim to explore this strategy in the context of line integrals. We will find that it does not always succeed but will find conditions where it does and how to use these to evaluate line integrals

A line integral involves

1) a vector field

$$\vec{v} = V_x(x,y,z)\,\hat{x} + V_y(x,y,z)\,\hat{y} + V_z(x,y,z)\,\hat{z}$$



2) a path described via a parameter, t

$x(t), y(t), z(t)$

Then

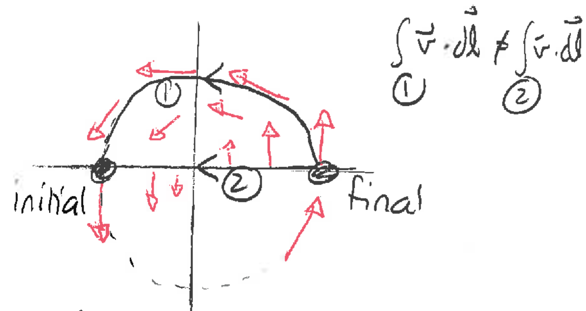
> The line integral of $\vec{v}$ along the path is
>
> $$\int \vec{v}\cdot d\vec{l} = \int_{t_i}^{t_f} \left[V_x \frac{dx}{dt} + V_y \frac{dy}{dt} + V_z \frac{dz}{dt} \right] dt$$
>
> where $V_x \equiv V_x(x(t), y(t), z(t))$, etc,.. are functions of t.

We then ask if a version of the fundamental theorem exists:

$$\int_{\text{path}} \vec{v}\cdot d\vec{l} = \text{some function of } x,y,z \text{ evaluated at } x(t_f), y(t_f), z(t_f) - \underline{\text{same}} \text{ function of } x,y,z \text{ evaluated at } t_i \ldots$$

Immediately we see that the r.h.side does not refer to the actual path from initial to final point. But we know that line integrals do sometimes depend on the path. Separately if this were true then the line integral of $\vec{v}$ around any closed loop

$$\int \vec{v}\cdot d\vec{l} = 0$$



and we can see that this is not always true either.

Fundamental theorem for gradients

There is one special situation where we can demonstrate a version of the fundamental theorem. Suppose that the vector field is a gradient of a scalar function. So for some $g = g(x,y,z)$,

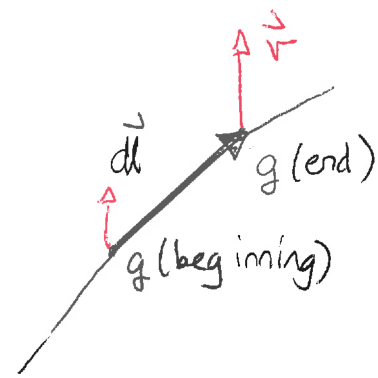
$$\vec{v} = \vec{\nabla} g$$

Then

$$\vec{v} = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z}$$

and along a line segment

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$



we get, along any segment

$$\vec{v}\cdot\vec{dl} = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz$$

$$= dg$$

$$= g(\text{end segment}) - g(\text{beginning segment})$$

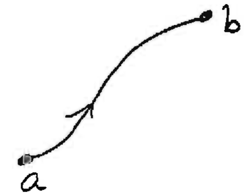
We can then add along all segments and then we will find

$$\sum \vec{v}\cdot\vec{dl} = g(\text{end path}) - g(\text{beginning path})$$

We arrive at the fundamental theorem for gradients.

Given a vector field $\vec{v}$ then there exists a function $g(x,y,z)$ such that $\vec{v} = \vec{\nabla} g$ if and only if for any path

$$\int_{\text{path}} \vec{v} \cdot \vec{dl} = \int_{\text{path}} \vec{\nabla} g \cdot \vec{dl} = g(b) - g(a)$$



where $g(b) = g(x,y,z)\Big|_{\text{evaluate at point } b}$ and b is the point at the end of the path. Similarly for a

Proof Suppose that $\vec{v} = \vec{\nabla} g$ and the path is $(x(t), y(t), z(t))$ with

$$a \equiv x(t_i), y(t_i), z(t_i)$$

$$b \equiv x(t_f), y(t_f), z(t_f)$$

Then

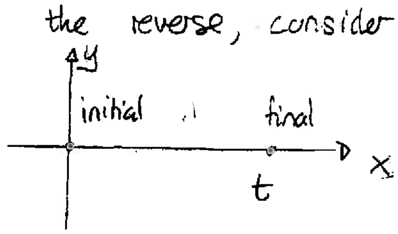
$$\int_a^b \vec{v} \cdot \vec{dl} = \int_a^b \vec{\nabla} g \cdot \vec{dl} = \int_{t_i}^{t_f} \left[\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} \right] dt$$

$$= \int_{t_i}^{t_f} \frac{d}{dt}\, g(x(t), y(t), z(t)) \; dt$$

$$= g(x(t_f), y(t_f), z(t_f)) - g(x(t_i), y(t_i), z(t_i))$$

$$= g(b) - g(a)$$

To see the reverse, consider the path



$\rightsquigarrow$ $x(s) = s \qquad 0 \le s \le t$

$y(s) = 0$

$z(s) = 0$

Then:

$$\int \vec{v} \cdot d\vec{l} = g(x(t), y(t), z(t)) - g(0,0,0)$$

$$\Rightarrow \int_0^t \left[V_x \frac{dx}{ds} + V_y \frac{dy}{ds} + V_z \frac{dz}{ds} \right] ds = g(x(t), y(t), z(t)) - g(0,0,0)$$

$$\Rightarrow \int_0^t V_x(x(s), y(s), z(s))\, ds = g(x(t), y(t), z(t)) - g(0,0,0)$$

Differentiate w.r.t. t.

$$\frac{d}{dt} \int_0^t \dots ds = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\cancel{\frac{dy}{dt}} + \frac{\partial g}{\partial z}\cancel{\frac{dz}{dt}}$$

$$\Rightarrow V_x(x(t), y(t), z(t)) = \frac{\partial g}{\partial x}$$

This gives the result for the illustrated point $x = t, y=0, z=0$. A similar construction could be extended to any point and any component. ∎

Consequences of the fundamental theorem for gradients are:

1) $\boxed{\text{If } \vec{v} = \vec{\nabla} g \text{ then } \int \vec{v}\cdot d\vec{l} \text{ only depends on initial and final points of the path}}$

2) If $\vec{v} = \vec{\nabla} g$ then around any closed loop $\oint_{loop} \vec{v} \cdot d\vec{l} = 0$

If around any closed loop $\oint_{loop} \vec{v}\cdot d\vec{l} = 0$ then $\vec{v} = \vec{\nabla} g$ for some g.

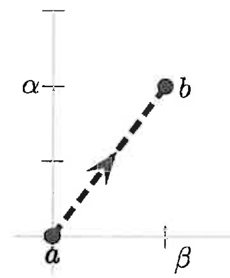
1 Fundamental theorem for gradients

Let $g(x,y,z) = \tau x^2 y$ where τ is constant.

a) Determine $\mathbf{v} := \boldsymbol{\nabla} g$.

b) Determine $\int \mathbf{v} \cdot \mathrm{d}\mathbf{l}$ along the path.

c) Determine $g(b) - g(a)$ and check that the fundamental theorem for gradients is valid.



Answer: a)

$$\vec{\nabla} g = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z}$$

$$= 2\tau x y\,\hat{x} + \tau x^2\hat{y} = \vec{v}$$

b) Along path

$$0 \leqslant x \leqslant \beta$$

$$y = \frac{\alpha}{\beta}x$$

So

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} = \left[\hat{x} + \frac{dy}{dx}\hat{y}\right]dx$$

$$= \left(\hat{x} + \frac{\alpha}{\beta}\hat{y}\right)dx$$

and

$$\vec{v}\cdot\vec{dl} = (v_x\hat{x} + v_y\hat{y})\cdot\left(\hat{x} + \frac{\alpha}{\beta}\hat{y}\right)dx$$

$$= \left(2\tau xy\,\hat{x} + \tau x^2\hat{y}\right)\left(\hat{x} + \frac{\alpha}{\beta}\hat{y}\right)dx$$

$$= \tau\left[2xy + x^2\frac{\alpha}{\beta}\right]dx$$

$$= \tau\left[2x\frac{\alpha}{\beta}x + x^2\frac{\alpha}{\beta}\right]dx = \frac{3\alpha}{\beta}x^2 dx$$

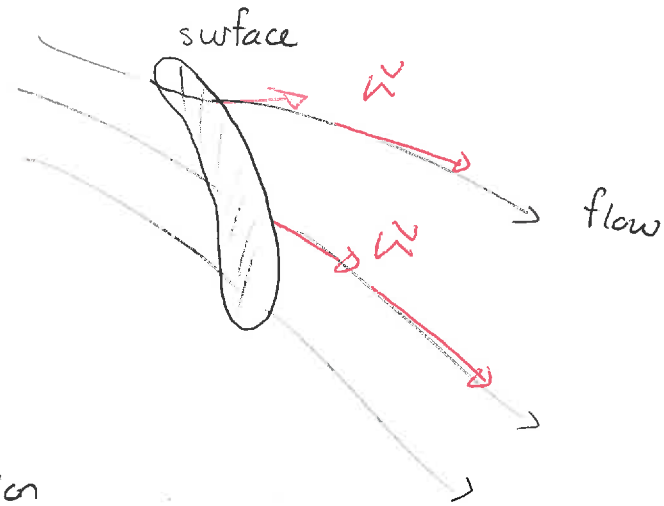
$$\int \vec{v}\cdot\vec{dl} = \tau\int_0^\beta \frac{3\alpha}{\beta}x^2\,dx = \tau\frac{\alpha}{\beta}x^3\Big|_0^\beta = \tau\alpha\beta^2$$

c) $g(b) = \tau\beta^2\alpha$ and $g(a) = 0$ $\Rightarrow g(b) - g(a) = \tau\alpha\beta^2 = \int \vec{v}\cdot\vec{dl}$

Surface integrals

We will also need to develop the notion of a flow or flux associated with a vector field. We illustrate this via fluid flow.

Example: Consider a fluid that flows in three dimensions. We would like to quantify the rate at which fluid volume passes through any imaginary surface. Specifically we want the volume of fluid that passes every second. This is called the (volume) flux of the fluid. This depends on



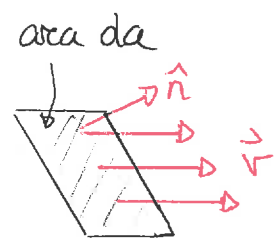
1) the surface – size, shape, orientation
2) the way in which the fluid flows.

The fluid flow is described via a velocity vector at each location

$\vec{v}$ = velocity vector field

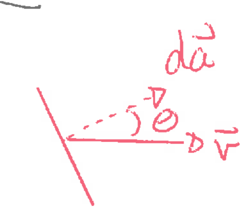
The calculation can be simplified by considering a small flat, planar fragment of surface across which the velocity vector is uniform

The shaded volume will all pass in time dt.



This volume is

$$dV = v\,dt\,da\cos\theta$$



and the volume rate of flow is

$$\frac{dV}{dt} = v\cos\theta\,da$$

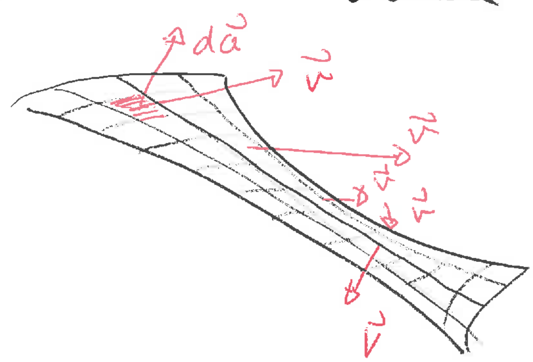
We define the area vector as

$$d\vec{a} = da\,\hat{n}$$

where $\hat{n}$ is the outward normal to the surface. Then, for this portion the volume flow rate is:

$$\vec{v} \cdot d\vec{a}$$

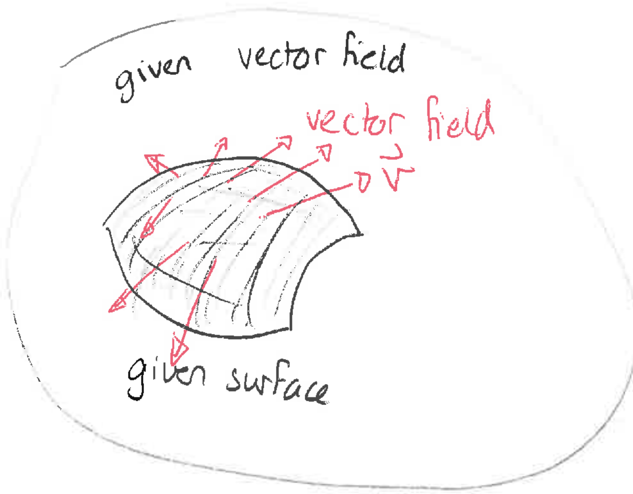
For a non-uniform velocity vector field and/or a curved surface we can break the surface into segments. The total flow rate is



$$\frac{dV}{dt} = \sum_{\text{all segments}} \vec{v} \cdot d\vec{a} = \int_{\text{surface}} \vec{v} \cdot d\vec{a}$$

as $da \to 0$

This is the idea behind a surface integral. It can be defined more precisely (see following pages)



surface integral

$$\int \vec{v} \cdot d\vec{a}$$

is a scalar

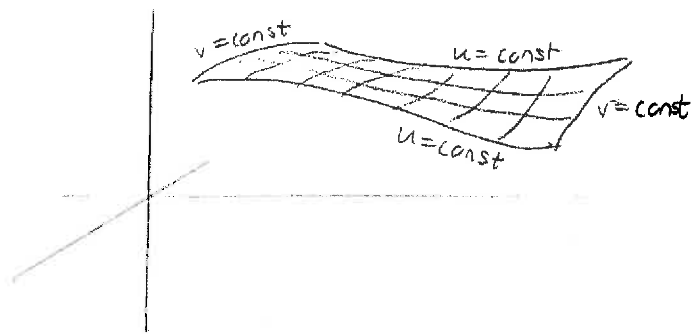
General construction of a surface integral

The general construction of a surface integral involves:

1) parameterizing the surface
2) constructing area vectors
3) constructing integrals

First we <u>parameterize the surface</u>. Any two dimensional surface in $\mathbb{R}^3$ can be parameterized by two scalar parameters u, v. Then the surface is described by

$$x(u,v),\ y(u,v),\ z(u,v)$$



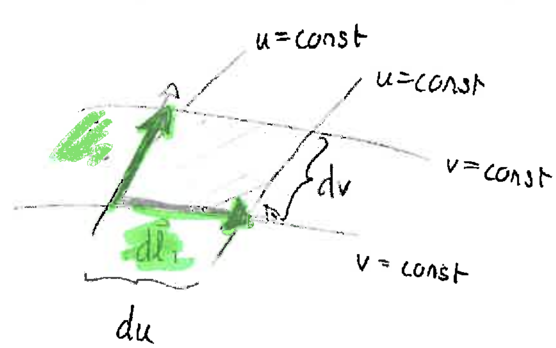
and u, v constitute a grid on the surface. It follows that, for any vector field

$$\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$$

this becomes a field that depends on the two parameters u, v.

$$\vec{F} = F_x(x(u,v), y(u,v), z(u,v))\,\hat{x} + F_y(x(u,v), y(u,v), z(u,v))\,\hat{y} + \dots$$

Now we <u>construct area vectors</u>, starting with the u,v grid. Using the illustration we see that the two vectors $\vec{dl}_1$ and $\vec{dl}_2$ can be used to construct a normal via



$$\vec{dl}_1 \times \vec{dl}_2$$

and this has area $|\vec{dl}_1 \times \vec{dl}_2|$

Now

$$\vec{dl}_1 = \left[\frac{\partial x}{\partial u}\hat{x} + \frac{\partial y}{\partial u}\hat{y} + \frac{\partial z}{\partial u}\hat{z}\right] du$$

$$\vec{dl}_2 = \left[\frac{\partial x}{\partial v}\hat{x} + \frac{\partial y}{\partial v}\hat{y} + \frac{\partial z}{\partial v}\hat{z}\right] dv$$

Then the area vector for this segment is

$$d\vec{a} = \vec{dl}_1 \times \vec{dl}_2$$

$$= \left[\frac{\partial x}{\partial u}\hat{x} + \frac{\partial y}{\partial u}\hat{y} + \frac{\partial z}{\partial u}\hat{z}\right] \times \left[\frac{\partial x}{\partial v}\hat{x} + \frac{\partial x}{\partial v}\hat{y} + \frac{\partial z}{\partial v}\hat{z}\right] du\, dv$$

Finally we construct the surface integral

$$\int_{\text{surface}} \vec{F} \cdot d\vec{a} \;:=\; \int \vec{F} \cdot (\vec{dl}_1 \times \vec{dl}_2)$$

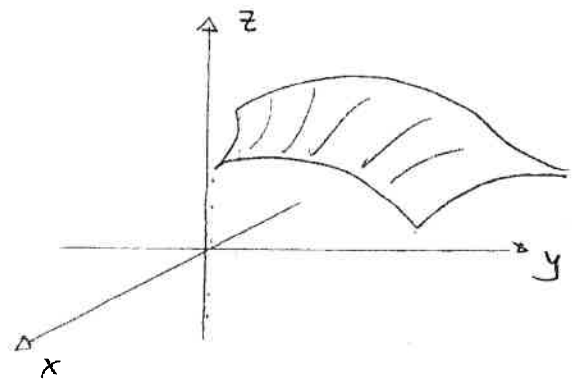
$$\equiv \iint_{\substack{u,v \\ \text{for} \\ \text{surface}}} \left\{ F_x \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v} \right) + F_y \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} \right) + F_z \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \right\} du\, dv$$

and this is an ordinary two dimensional integral

Surfaces described in terms of usual co-ordinates

Suppose that the surface projects onto the x,y plane in a 1-1 fashion. Then for some function $g(x,y)$ the surface is described via

$$z = g(x,y)$$



We can use parameters

$$u = x$$
$$v = y$$

In this case the area vector becomes

$$d\vec{a} = \left(\cancelto{1}{\frac{\partial x}{\partial x}}\hat{x} + \cancelto{0}{\frac{\partial y}{\partial x}}\hat{y} + \frac{\partial z}{\partial x}\hat{z}\right) \times \left(\cancelto{0}{\frac{\partial x}{\partial y}}\hat{x} + \cancelto{1}{\frac{\partial y}{\partial y}}\hat{y} + \frac{\partial z}{\partial y}\hat{z}\right) dx\,dy$$

$$= \left(\hat{x} + \frac{\partial g}{\partial x}\hat{z}\right) \times \left(\hat{y} + \frac{\partial g}{\partial y}\hat{z}\right) dx\,dy$$

$$= \left(\hat{z} - \frac{\partial g}{\partial x}\hat{x} - \frac{\partial g}{\partial y}\hat{y}\right) dx\,dy$$

$$= -\left[\frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} - \hat{z}\right] dx\,dy$$

For example on a hemisphere of radius r $\quad z = \underbrace{\sqrt{r^2 - x^2 - y^2}}_{g(x,y)}$

and

$$d\vec{a} = -\left[\frac{-x}{\sqrt{r^2-x^2-y^2}}\hat{x} - \frac{y}{\underbrace{\sqrt{r^2-x^2-y^2}}_{=z}}\hat{y} - \hat{z}\right] dx\,dy$$

$$= \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{r^2-x^2-y^2}}\, dx\,dy$$

and this points radially outwards