Fri: HW 5pm

Mon: Read 1.3.3 gradients, 1.3.2

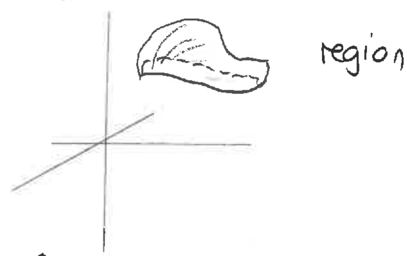
Tues: HW 4 due

Charge + charge density

Recall that the charge density $\rho(x,y,z)$ is

- a function of location with units C/m³

and is defined s.t.

> The charge in any region is
>
> $$Q = \iiint_{\text{region}} \rho(x,y,z)\, dx\, dy\, dz$$
>
> 
>
>
> where the limits of integration are arranged so as to encapsulate the entire region

Note that:

1) the charge density depends on location so it is a scalar function of x, y, z

2) the total charge inside any region does not depend on particular locations (x, y, z) within the region. This is a scalar but not a scalar function of x, y, z

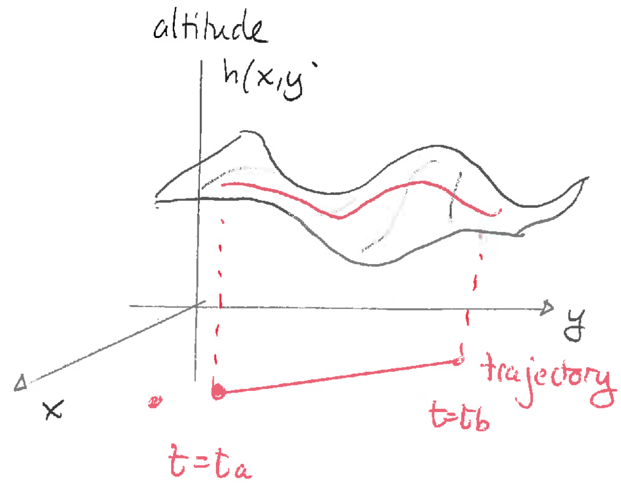
We often write $Q = \int \rho \, d\tau$

→ $d\tau$ does NOT mean integrate w.r.t. variable τ

→ DOES mean integrate w.r.t. x, y, z

→ ρ usually not constant, depends on x, y, z

Line Integrals



Imagine a journey across a two dimensional surface, We might want to know the average altitude during the journey. This require:

1) a description of the landscape via an altitude function $h(x,y)$
2) a description of the trajectory followed – this is a path in x,y space represented as

$$x(t), y(t)$$

At each location along the path, parameterized by t, we could compute the altitude:

$$\text{altitude at } t = h(x(t), y(t))$$

This is a function of one variable, t. Then the average height would be

$$\bar{h} = \frac{1}{\text{total distance}} \int_{t_a}^{t_b} \underbrace{h(x(t), y(t))}_{\text{only depends on } t} dt$$

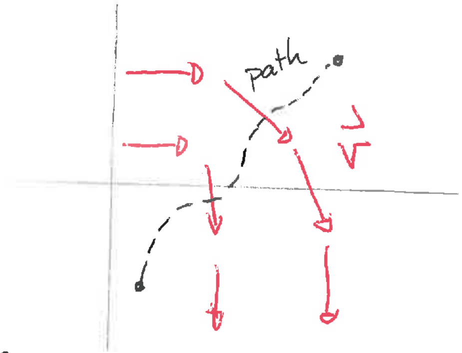
This is an example of a line integral of a scalar function and is the basis for line integrals of vector fields. For the line integral of a vector field, we need:

1) a vector field $\vec{v} = V_x(x,y,z)\hat{x} + V_y(x,y,z)\hat{y} + V_z(x,y,z)\hat{z}$

2) a path /curve $x(t), y(t), z(t)$

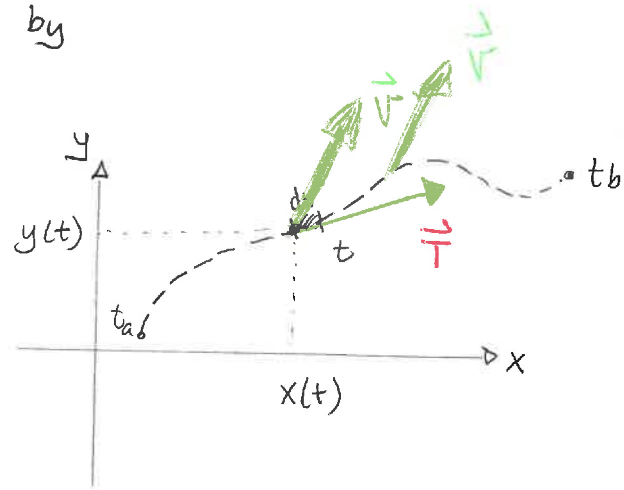
There are potentially various possibilities. We explore one in which we combine the field and curve to produce a one dimensional integral.

We will have to show that whatever definition we provide the result only depends on the vector field and the curve and not on the basis or path parametrization



An example from mechanics is suggestive. The work done by any force can be computed by

$$W = \sum \underbrace{\vec{F}}_{\text{force}} \cdot \underbrace{\Delta\vec{r}}_{\text{tangent to path.}}$$



More formally suppose that the path is parameterized by $x(t), y(t), z(t)$. Then the tangent to the curve is

$$\vec{T} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

Now, given a separate vector field $\vec{v}$ we can determine the component of $\vec{v}$ along $\vec{T}$ via $\vec{v}\cdot\vec{T}\,dt$. We can add these to get

$$\sum \vec{v}\cdot\vec{T}\,dt$$

The actual sum can be performed via

$$\vec{v}\cdot\vec{T} = \underbrace{V_x(\underbrace{x(t), y(t), z(t)}_{\text{location at } t})}_{x \text{ comp of } \vec{v}}\frac{dx}{dt} + V_y(x(t), y(t), z(t))\frac{dy}{dt} + V_z(x(t), y(t), z(t))\frac{dz}{dt}$$

only depends on t

Then we define:

> The line integral of $\vec{v}$ along the curve $x(t), y(t), z(t)$ from $t = t_a$ to $t = t_b$ is:
>
> $$\int \vec{v} \cdot \vec{T}\, dt = \int_{t_a}^{t_b} \left[V_x\big(\underbrace{x(t), y(t), z(t)}\big) \frac{dx}{dt} + V_y \frac{dy}{dt} + V_z \frac{dz}{dt} \right] dt$$
>
> also appears here

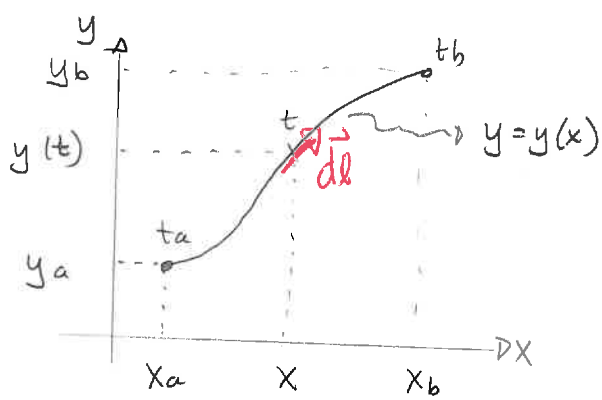
One can prove:

1) the integral delivers the same result regardless of how the path is parameterized.

2) the integral is independent of the co-ordinate system and the basis vectors that yield components V_x, V_y, V_z.

3) the integral can depend on the path between the endpoints

Sometimes we can use co-ordinates to parameterize the curve. Suppose each value of x only yields one value of y, z. Then



$$\int \vec{v} \cdot \vec{T}\, dt = \int_{x_a}^{x_b} \left[V_x + V_y \frac{dy}{dx} + V_z \frac{dz}{dx} \right] dx$$

which requires $y = y(x)$ and $z = z(x)$ for the curve. Note that we can use a line element vector

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$= \left[\hat{x} + \frac{dy}{dx}\hat{y} + \frac{dz}{dx}\hat{z} \right] dx$$

Then

$$\vec{v} \cdot \vec{dl} = \left[v_x \hat{x} + v_y \hat{y} + v_z \hat{z}\right]\left[\hat{x} + \frac{dy}{dx}\hat{y} + \frac{dz}{dx}\hat{z}\right] dx$$

$$= \left[v_x + v_y \frac{dy}{dx} + v_z \frac{dz}{dx}\right] dx$$

and we then can write the line integral as

$$\int \vec{v} \cdot \vec{dl}$$

This also allows a more generic notation:

$$\vec{v} \cdot \vec{dl} = v_x dx + v_y dy + v_z dz$$

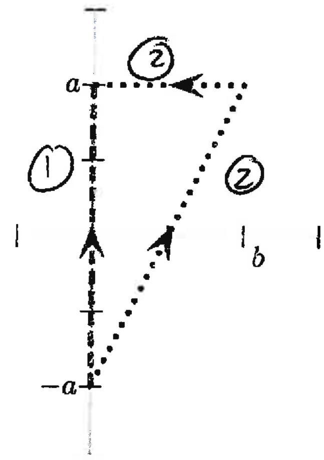
and this can be adapted so that any of x, y, z would be a parameter.

1 **Line integral**

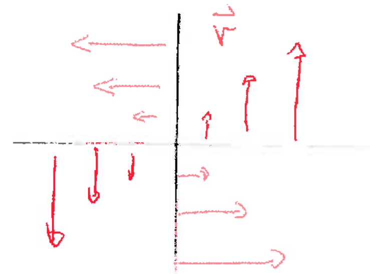
Let $\mathbf{v} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$. Determine

$$\int \mathbf{v} \cdot d\mathbf{l}$$

along each of the two paths.



Along path 1



$$x = 0$$
$$-a \le y \le a$$

So make y the parameter.

$$\vec{v} \cdot \vec{dl} = v_x\, dx + v_y\, dy$$

parameter

$$= v_x \frac{dx}{dy} dy + v_y\, dy = \left[v_x \frac{dx}{dy} + v_y\right] dy$$

Now along this path $x = \text{const} \Rightarrow \frac{dx}{dy} = 0$. Also $v_y = x = 0$ along this path so

$$\vec{v} \cdot \vec{dl} = [v_x 0 + 0]\, dy = 0 \quad \Rightarrow \quad \vec{v} \cdot \vec{dl} = 0 \Rightarrow \int_{\text{path}\,1} \vec{v} \cdot \vec{dl} = 0$$

We see that everywhere along the path $\vec{v} \cdot \vec{dl} = 0$ by the sketch

Along path ② The path has a c.c.w sense as does $\vec{v}$. So we expect $\int \vec{v}\cdot\vec{dl} > 0$

Diagonal section: $0 \le x \le b$ ← parameter

$$y = \frac{2a}{b}x - a$$

$$\vec{v}\cdot\vec{dl} = v_x dx + v_y dy$$

$$= v_x\, dx + v_y \frac{dy}{dx} dx$$

$$= \left[v_x + v_y \frac{dy}{dx}\right] dx$$

Here $v_x = -y = -\left(\frac{2a}{b}x - a\right) = a - \frac{2a}{b}x$

$v_y = x$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2a}{b}x - a\right) = \frac{2a}{b}$

So $\vec{v}\cdot\vec{dl} = \underbrace{a - \frac{2a}{b}x}_{v_x} + \underbrace{x}_{v_y}\,\underbrace{\frac{2a}{b}}_{dy/dx} = a$

So $\int_{diag} \vec{v}\cdot\vec{dl} = \int_0^b a\,dx = ab \Rightarrow \int_{diag} \vec{v}\cdot\vec{dl} = ab$

Horizontal section The direction is complicated. We note that

$$\overset{\longleftarrow}{\int_{\text{path}} \vec{v}\cdot d\vec{l}} = - \overset{\longrightarrow}{\int_{\text{path}} \vec{v}\cdot d\vec{l}}$$

So

$$\int_{\substack{\text{horiz} \\ \text{right to left}}} \vec{v}\cdot d\vec{l} = - \int_{\substack{\text{horiz} \\ \text{left to right}}} \vec{v}\cdot d\vec{l}$$

For left to right $0 \leq x \leq b$ ← parameter

$y = a$

$$\vec{v}\cdot d\vec{l} = V_x\,dx + V_y\,dy$$

$$= V_x\,dx + V_y \frac{dy}{dx}\,dx = \left[V_x + V_y \frac{dy}{dx}\right] dx$$

Here $V_x = -y = -a$

$V_y = x =$ variable.

$\frac{dy}{dx} = 0$ since y is constant

$$\int_{\text{left to right}} \vec{v}\cdot d\vec{l} = \int_0^b -a\,dx = -ab$$

Thus $\int_{\substack{\text{horiz} \\ r \text{ to } l}} \vec{v}\cdot d\vec{l} = ab$

Adding gives $\int_{①} \vec{v}\cdot d\vec{l} = ab + ab = \int_{①} \vec{v}\cdot d\vec{l} = 2ab$

Note that, if for the horiz section

$$\vec{dl} = - dx\, \hat{x}$$

and we integrate from $x=b$ to $x=a$ we get

$$\int \vec{v} \cdot \vec{dl} = -\int_b^0 v_x\, dx = \int_b^0 y\, dx = a \int_b^0 dx = -ab$$

which is Incorrect.

The example illustrates the fact that

Line integrals generally depend on the path between initial + final locations