Phys 311 2020

Lecture 5.

Fri: HW 3 by ~~6~~ 5pm

Read 1.3.1

Vector derivatives of sums + products 1.2.6

We can differentiate sums and products. The various derivatives are always linear:

Gradient:	$\vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g$
Divergence:	$\vec{\nabla}\cdot(\vec{u}+\vec{v}) = \vec{\nabla}\cdot\vec{u} + \vec{\nabla}\cdot\vec{v}$
Curl:	$\vec{\nabla}\times(\vec{u}+\vec{v}) = \vec{\nabla}\times\vec{u} + \vec{\nabla}\times\vec{v}$

Rules involving multiplication can be derived from basis product rules of differentiation.

1) gradient

$$\boxed{\begin{aligned}\vec{\nabla}(fg) &= f\vec{\nabla}g + g\vec{\nabla}f \\ \vec{\nabla}(\vec{A}\cdot\vec{B}) &= \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}\end{aligned}}$$

Note that $(\vec{A}\cdot\vec{\nabla})\vec{B} \neq (\vec{\nabla}\cdot\vec{A})\vec{B}$. What this actually means is

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = \left[A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right]\left[B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right]$$

$$= \left[A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right]\hat{x}$$

$$+ \left[A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right]\hat{y} + \left[A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right]\hat{z}$$

2) <u>divergence</u> One can show:

$$\vec{\nabla} \cdot (f\vec{A}) = (\vec{\nabla} f) \cdot \vec{A} + f(\vec{\nabla} \cdot \vec{A})$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

3) <u>curl</u>

Similar product rules exist for curl

↳ see pg 21 section 1.2.6

1 Divergence of a dot product

In general

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

a) Verify that this is true for

$$\mathbf{A} = x^2\hat{\mathbf{x}}$$
$$\mathbf{B} = \hat{\mathbf{x}}$$

b) Check that, for these vectors, $(\mathbf{B}\cdot\nabla)\mathbf{A} \neq (\nabla\cdot\mathbf{B})\mathbf{A}$.

Ans: a) LHS $\vec{A}\cdot\vec{B} = x^2 \Rightarrow \vec{\nabla}(\vec{A}\cdot\vec{B}) = 2x\,\hat{x}$

RHS $\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 0 & 0 \end{vmatrix} = \hat{x}\,0 + \hat{y}\underbrace{\frac{\partial}{\partial z}x^2}_{0} + \hat{z}\underbrace{\left(-\frac{\partial}{\partial y}x^2\right)}_{0}$

$\Rightarrow \vec{\nabla}\times\vec{A} = 0$

$\vec{\nabla}\times\vec{B} = 0$ since $\vec{B}$ is constant.

$(\vec{A}\cdot\vec{\nabla})\vec{B} = \left(x^2\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}\right)\vec{B} = x^2\frac{\partial\vec{B}}{\partial x} = 0$ since $\vec{B}$ constant

$(\vec{B}\cdot\vec{\nabla})\vec{A} = \left(1\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}\right)\vec{A} = \frac{\partial\vec{A}}{\partial x} = 2x\,\hat{x}$

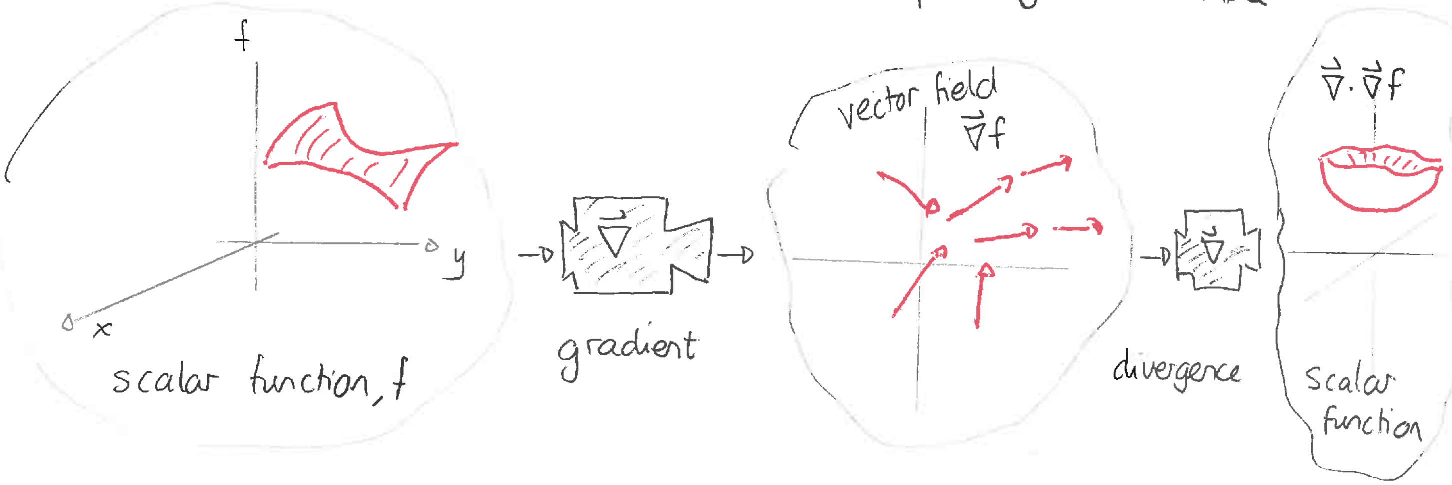
Thus RHS $= 2x\hat{x} =$ LHS ✓

b) $(\vec{\nabla}\cdot\vec{B})\vec{A} = \underbrace{(\vec{\nabla}\cdot\hat{x})}_{0}\,x^2\hat{x} = 0$

So $\underbrace{(\vec{\nabla}\cdot\vec{B})\vec{A}}_{0} \neq \underbrace{(\vec{B}\cdot\vec{\nabla})\vec{A}}_{2x\hat{x}}$

Multiple derivatives

Three dimensional differentiation can be done repeatedly. For example



In this example:

$$\vec{\nabla}\cdot\vec{\nabla} f = \vec{\nabla}\cdot\left[\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right]$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is a common operation in electromagnetic theory (in the context of electric potential), other branches of physics and also mathematics. Thus we define

> The Laplacian operator, $\vec{\nabla}^2$, maps functions to functions via
>
> $$f \xrightarrow{\vec{\nabla}^2} \vec{\nabla}^2 f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We will eventually see that

$$\vec{\nabla}^2 V = \text{function of charge distribution}$$

2 Multiple derivatives in three dimensions

Let

$$f(x,y,z) = \frac{x^2+y^2}{4}.$$

a) Determine $\nabla^2 f$.

b) Determine $\nabla \times (\nabla f)$.

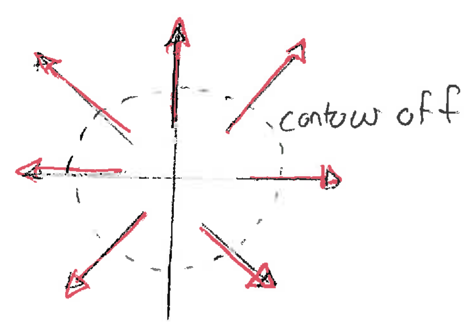
Ans: a) $$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}\left(\frac{x^2}{4}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{y^2}{4}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

b) $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$= \frac{x}{2}\hat{x} + \frac{y}{2}\hat{y}$$

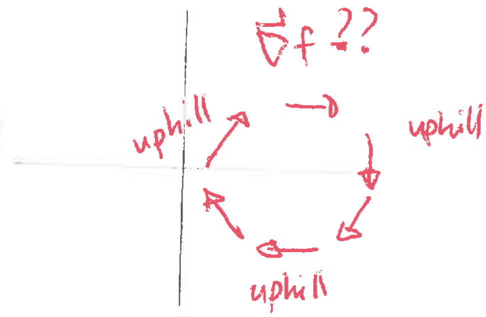


So $$\vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{2} & \frac{y}{2} & 0 \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z}\frac{y}{2}\right) + \hat{y}\left(\frac{\partial}{\partial z}\frac{x}{2} - \frac{\partial}{\partial x} 0\right) + \hat{z}\left(\frac{\partial}{\partial x}\frac{y}{2} - \frac{\partial}{\partial y}\frac{x}{2}\right)$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} f) = 0$$

The example illustrates a general rule about gradients. A gradient cannot circle as that would indicate a closed loop along which the function keeps increasing.

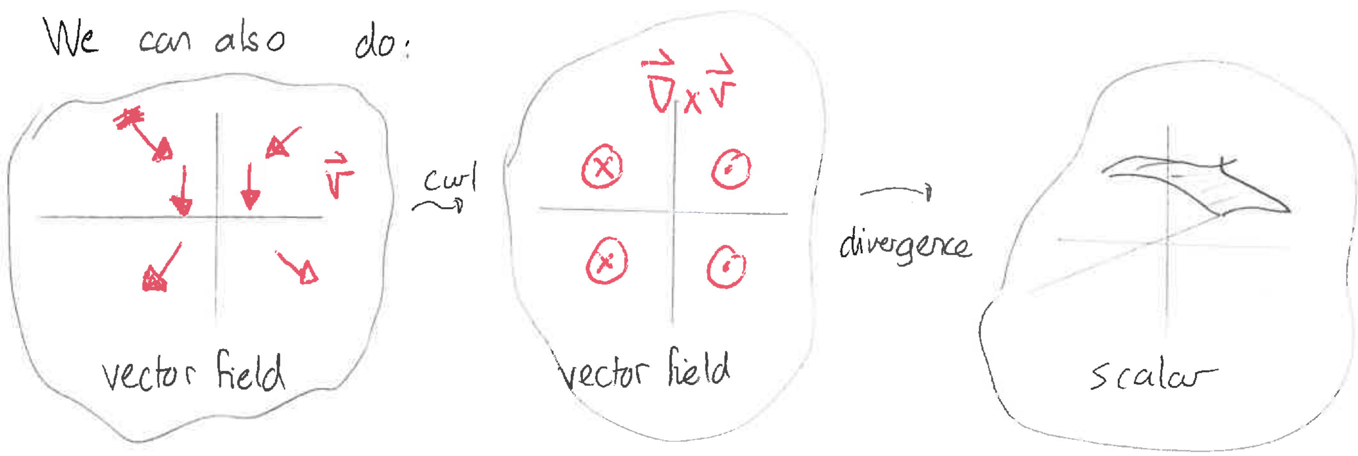
One can show using standard calculus that



For any suitably differentiable function, f,

$$\vec{\nabla} \times \vec{\nabla} f = 0$$

We can also do:



Again we can show generally.

For any suitably differentiable vector field, $\vec{v}$,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$$

Integration of functions in three dimensions

We will encounter integrals of the following types in electromagnetic theory

1) integrals of scalar functions
2) integrals of vector fields — line integrals and surface integrals

In all cases the basic concepts and theorems use the fundamental theorem of calculus:

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

Volume integrals of scalar functions

Consider a scalar function of three position variables. In electromagnetic theory a common example is the volume charge density which describes how charge is distributed in space.

The volume charge density is a function $\rho(x,y,z)$ with units of C/m^3 which has the approximate meaning:

The charge contained in a region

$x \to x+dx$

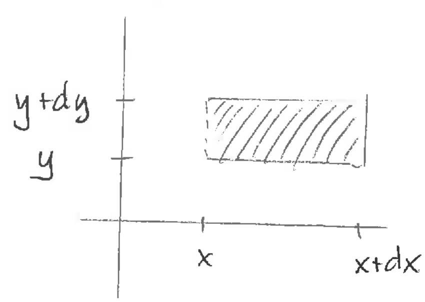
$y \to y+dy$

$z \to z+dz$

is

$$\underbrace{\rho(x,y,z)}_{\text{density}}\overbrace{dx\,dy\,dz}^{\text{volume region}}$$

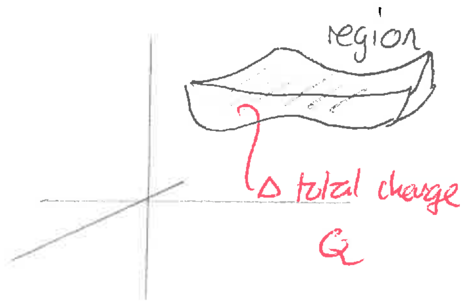
as $dx, dy, dz \to 0$



A more precise meaning is:

Consider any region of space. Then the volume charge density is the function $\rho(x,y,z)$ with units C/m^3 such that the total charge contained in this region is:

$$Q = \iiint_{\text{region}} \rho(x,y,z)\, dx\,dy\,dz$$



A frequent short notation is that the volume $dx\,dy\,dz$ is abbreviated as

$$d\tau = dx\,dy\,dz$$

and the charge in the region is

$$Q = \int_{\text{region}} \rho\, d\tau$$

Note that this does not mean

$$Q = \int \rho\, d\tau$$

DOES NOT mean integrate wrt variable τ.

DOES mean integrate w.r.t $x, y,$ and z

USUALLY is not constant and cannot be pulled out of the integral

USUALLY depends on x, y, z.

3 Three dimensional charge distribution

Within the region $-a \leqslant x, y, z \leqslant a$ with $a > 0$ the charge density is

$$\rho(x, y, z) = \frac{4q}{a^5}\, xy$$

where q is a constant with dimensions of charge.

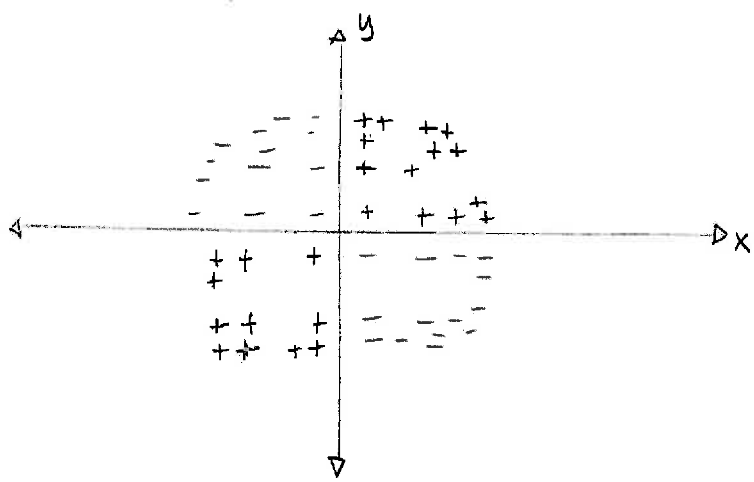
a) In the region $-a \leqslant xy \leqslant a$ in the xy plane sketch whether the charge is positive or negative indicating regions with greater and smaller charge density.

b) Determine the total charge in the region $-a \leqslant x, y, z \leqslant a$.

c) Determine the total charge in the region $0 \leqslant x, y, z \leqslant a$.

d) Determine the total charge contained in the segment of a sphere of radius a in the quadrant where $0 \leqslant x, y, z \leqslant a$.

Answer: a)



b) region

$$-a \leq x \leq a$$
$$-a \leq y \leq a$$
$$-a \leq z \leq a$$

$$Q = \int_{-a}^{a} dz \int_{-a}^{a} dy \int_{-a}^{a} dx\, \rho(x,y,z)$$

$$= \int_{-a}^{a} dz \int_{a}^{a} dy \int_{-a}^{a} \frac{4q}{a^5} xy$$

$$\Rightarrow \quad Q = \frac{4q}{a^5}\int_{-a}^{a} dz \int_{-a}^{a} y\,dy \int_{-a}^{a} x\,dx$$

$$= \frac{4q}{a^5}\, 2a \underbrace{\frac{y^2}{2}\Big|_{-a}^{a}}_{0} \underbrace{\frac{x^2}{2}\Big|_{-a}^{a}}_{0} \quad \Rightarrow \quad Q = 0$$

c) Now

$$0 \leq x \leq a$$
$$0 \leq y \leq a$$
$$0 \leq z \leq a$$

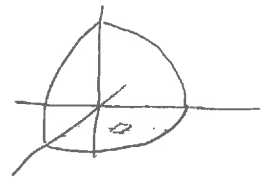
$$Q = \frac{4q}{a^5}\int_0^a dz \int_0^a y\,dy \int_0^a x\,dx$$

$$= \frac{4q}{a^5}\left[a\right]\left[\frac{y^2}{2}\Big|_0^a\right]\left[\frac{x^2}{2}\Big|_0^a\right]$$

$$= \frac{4q}{a^5}\, a\, \frac{a^2}{2}\, \frac{a^2}{2}$$

$$\Rightarrow \quad Q = q$$

d) Here



$$0 \le x \le a$$
$$0 \le y \le \sqrt{a^2-x^2}$$
$$0 \le z \le \sqrt{a^2-x^2-y^2}$$

So

$$Q = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} dy \int_0^{\sqrt{a^2-x^2-y^2}} dz \ \frac{4q}{a^5} xy$$

$$= \frac{4q}{a^5} \int_0^a dx\, x \int_0^{\sqrt{a^2-x^2}} dy\ y \sqrt{a^2-x^2-y^2}$$

$$= \frac{4q}{a^5} \int_0^a dx\ x \ (a^2-x^2-y^2)^{3/2} \frac{2}{3} \frac{1}{2} (-1) \Big|_0^{\sqrt{a^2-x^2}}$$

$$= -\frac{4q}{3a^5} \int_0^a dx\ x \left[0 - (a^2-x^2)^{3/2} \right]$$

$$= \frac{4q}{3a^5} \int_0^a x (a^2-x^2)^{3/2} dx$$

$$= \frac{4q}{3a^5} (a^2-x^2)^{5/2} \frac{2}{5} \left(-\frac{1}{2}\right) \Big|_0^a$$

$$= \frac{4q}{15a^5} (a^2)^{5/2} \quad \Rightarrow \quad Q = \frac{4}{15} q$$