Phys 311 2020

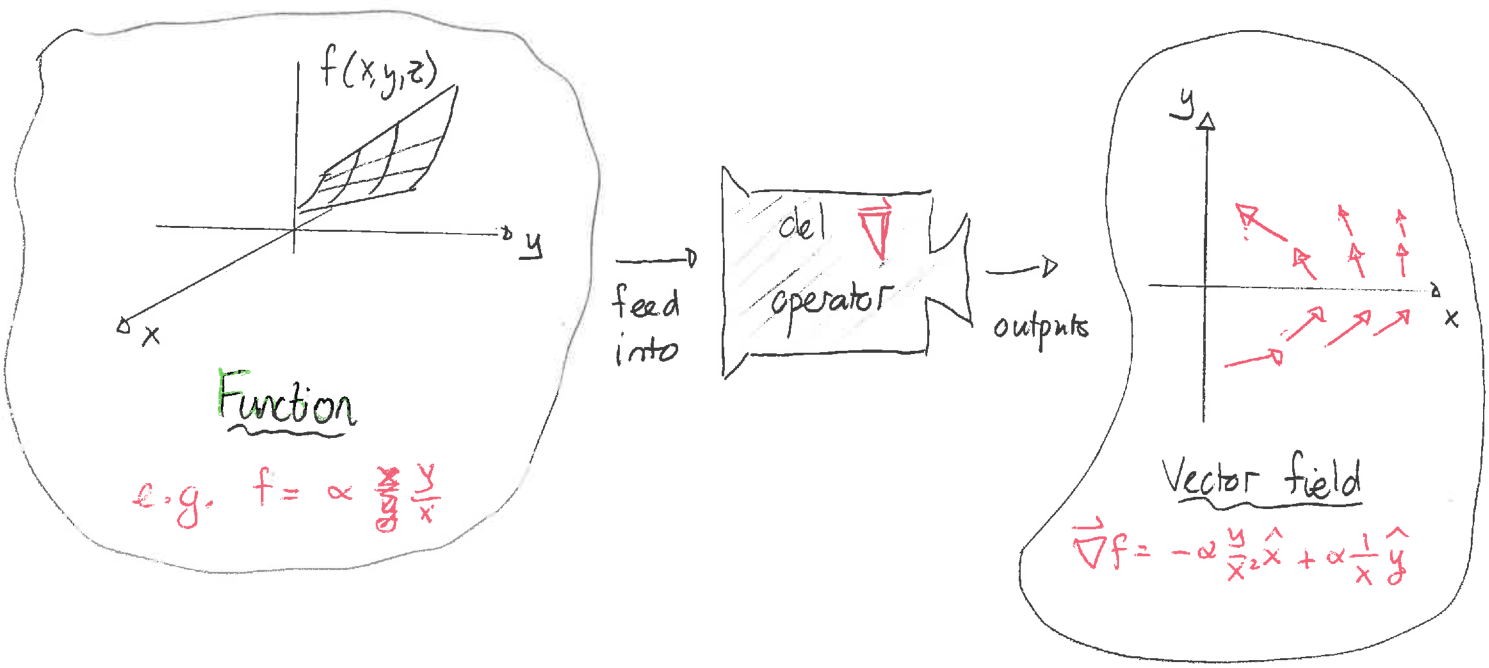
Lecture 4.

Tues: HW by 5pm

Weds: Read

Del Operator

The gradient is an operation that takes a function and produces a vector field. We can imagine that this is carried out by a piece of machinery or a mathematical entity called the "del" operator



We can imagine that the del operator has an existence independent of the input and the output in the same way as any other machine exists independent of these, e.g. printer exists independent of paper/data/output.

We write:

$$f(x,y,z) \xrightarrow{\vec{\nabla}} \vec{\nabla} f$$

or

$$\vec{\nabla}(\ldots) = \hat{x}\frac{\partial}{\partial x}(\ldots) + \hat{y}\frac{\partial}{\partial y}(\ldots) + \hat{z}\frac{\partial}{\partial z}(\ldots)$$

slots for input function

So we write

$$\boxed{\vec{\nabla} := \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.}$$

Notes:

1) the $\vec{\nabla}$ operator is not a conventional function as it contains derivative operators that await an input function.

2) the derivatives act on a function and not basis vectors, so

$$\frac{\partial}{\partial x}\hat{x} \neq 1 \text{ or } \vec{1}$$

3) the $\vec{\nabla}$ operator has some vector nature but is not a vector nor a vector field as we cannot provide components until we have supplied a function

4) when using $\vec{\nabla}$ do not forget the basis vectors. So

$$\vec{\nabla}f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} \Big] \text{ vector}$$

$$\neq \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \Big] \text{ function}$$

1 Del operator

a) Apply the del operator,

$$\boldsymbol{\nabla} := \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.$$

to the function

$$f(x,y) = \frac{1}{4}\,x^2y^2.$$

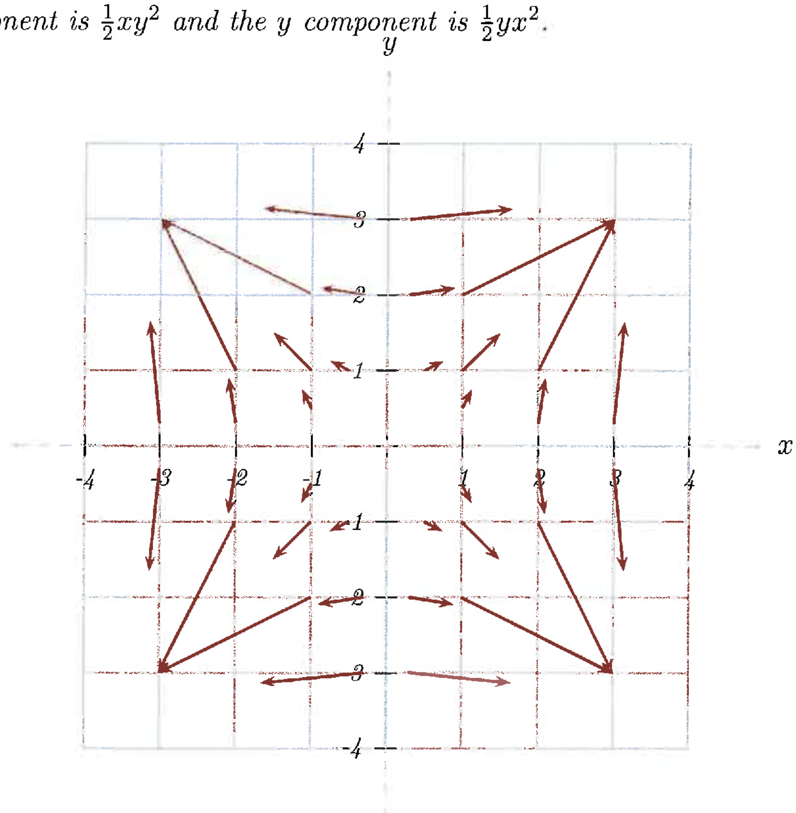
b) Provide expressions for the components that result from applying $\boldsymbol{\nabla}$ to f and sketch the resulting vector field.

Answer:

a)

$$\begin{aligned}\boldsymbol{\nabla} f &= \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \\ &= \frac{1}{2}xy^2\hat{\mathbf{x}} + \frac{1}{2}yx^2\hat{\mathbf{y}}.\end{aligned}$$

b) The x-component is $\frac{1}{2}xy^2$ and the y component is $\frac{1}{2}yx^2$.



Calculus on vector fields

We will need methods for differentiating and integrating vector fields. Differentiation involves comparing the vector field at one location to that at nearby locations. We need to first consider what can be compared. Generically any vector field can be expressed as:

$$\vec{V} = \vec{V}(\underbrace{x,y,z}_{\text{value of location}}) = V_x(\underbrace{x,y,z}_{\text{location}})\,\hat{x} + V_y(x,y,z)\,\hat{y} + V_z(x,y,z)\,\hat{z}$$

labels component

basis vector

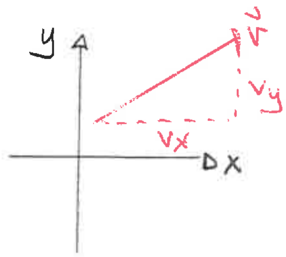
Any definition of a derivative of a vector field should be such that it yields the same outcome regardless of the basis used. We first provide a definition that DOES NOT do so

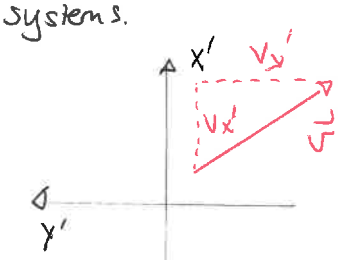
Example Suppose that we define a derivative of $\vec{V}$ via

BAD DEFINITION

$$\frac{\partial V_x}{\partial x}\hat{x} + \frac{\partial V_y}{\partial y}\hat{y} + \frac{\partial V_z}{\partial z}\hat{z}$$

where x, y, z is any Cartesian co-ordinate system. We consider calculation in two systems.





The same vector will have different components in the two frames.

Unprimed frame/basis — Primed frame/basis

Calculate

$$\underbrace{\frac{\partial v_x}{\partial x}\hat{x} + \frac{\partial v_y}{\partial y}\hat{y} + \frac{\partial v_z}{\partial z}\hat{z}}$$

SAME PROCEDURE

Calculate

$$\underbrace{\frac{\partial v_x'}{\partial x'}\hat{x}' + \frac{\partial v_y'}{\partial y'}\hat{y}' + \frac{\partial v_z'}{\partial z'}\hat{z}'}$$

ARE THESE SAME VECTOR?

For example if $\vec{v} = x\hat{x}$ then we need

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

$$\vec{v}' = v_x'\hat{x}' + v_y'\hat{y}' + v_z'\hat{z}'$$

Unprimed

$v_x = x$, $v_y = 0$, $v_z = 0$

Primed

$x' = y$ $\Leftrightarrow$ $\hat{y} = \hat{x}'$

$y' = -x$ $\quad$ $\hat{x} = -\hat{y}'$

$\Rightarrow \vec{v} = -y'(-\hat{y}') = y'\hat{y}$

$\Rightarrow v_x' = 0 \quad v_y' = y' \quad v_z' = 0$

So

$$\frac{\partial v_x}{\partial x}\hat{x} + \frac{\partial v_y}{\partial y}\hat{y} + \frac{\partial v_z}{\partial z}\hat{z}$$

$$= \hat{x}$$

$$\cancel{\frac{\partial v_x'}{\partial x'}}\hat{x}' + \frac{\partial v_y'}{\partial y'}\hat{y}' + \cancel{\frac{\partial v_z'}{\partial z'}}\hat{z}'$$

$$= \hat{y}' = -\hat{x}$$

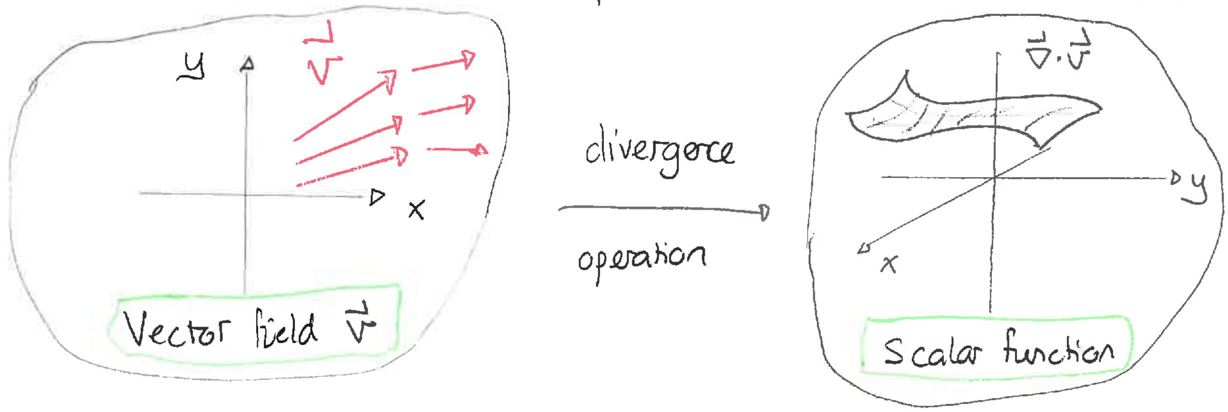
This definition of a derivative gives an outcome that depends on the basis. It is no good.

Divergence of a vector field

Suppose that, in a Cartesian co-ordinate system,

$$\vec{v} = V_x(x,y,z)\,\hat{z} + V_y(x,y,z)\,\hat{y} + V_z(x,y,z)\,\hat{z}$$

Then the divergence of $\vec{v}$ maps the vector field to a function



It is defined by

$$\boxed{\vec{\nabla}\cdot\vec{v} := \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}}$$

no vectors

This can be shown to be independent of Cartesian co-ord system

Notes:

1) the divergence of a vector is a scalar function and does NOT contain any basis vectors

2) the divergence satisfies

$$\vec{\nabla}\cdot(\vec{u}+\vec{v}) = \vec{\nabla}\cdot\vec{u} + \vec{\nabla}\cdot\vec{v}$$

3) We can use:

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}\right)$$

$$= \hat{x}\cdot\frac{\partial}{\partial x} v_x\hat{x} + \hat{x}\frac{\partial}{\partial x}\cdot v_y\hat{y} + \dots$$

$$= \frac{\partial v_x}{\partial x}\overset{1}{\cancel{\hat{x}\cdot\hat{x}}} + \frac{\partial v_y}{\partial x}\overset{0}{\cancel{\hat{x}\cdot\hat{y}}} + \dots$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

2 Divergence

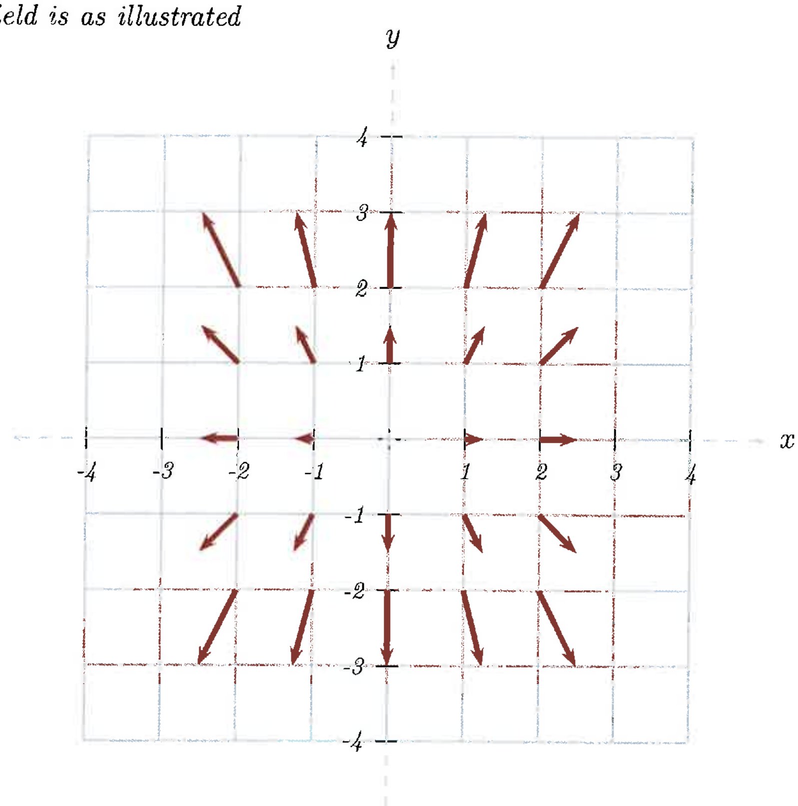
For each of the following, **sketch** the vector field and **determine its divergence.**

a) $\mathbf{v} = \frac{x}{4}\,\hat{\mathbf{x}} + \frac{y}{2}\,\hat{\mathbf{y}}$

b) $\mathbf{v} = \frac{-y}{2}\,\hat{\mathbf{x}} + \frac{x}{2}\,\hat{\mathbf{y}}$

Answer:

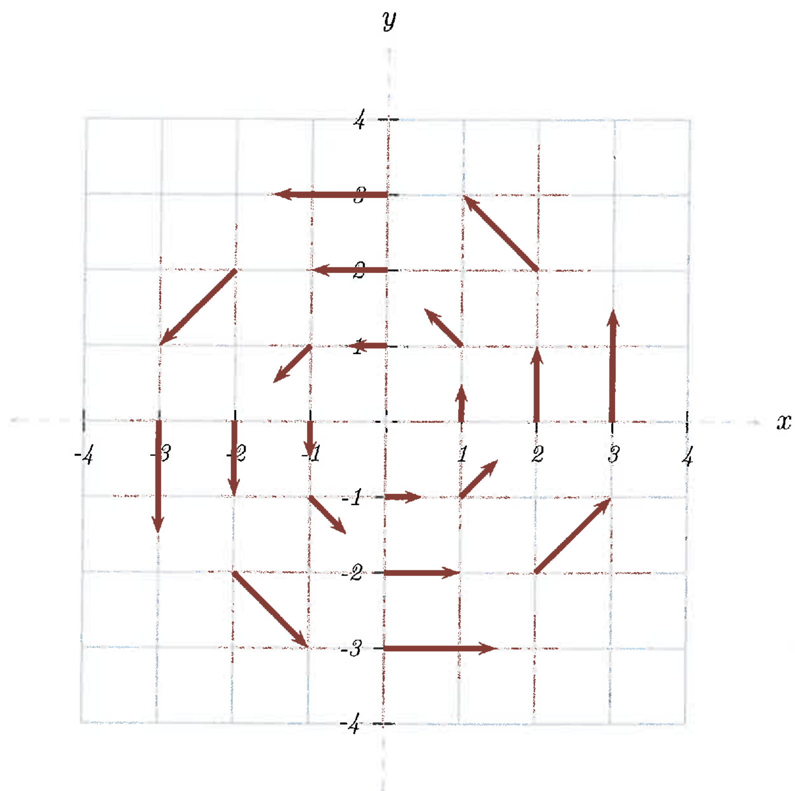
a) The vector field is as illustrated



The components are $v_x = x/4$ *and* $v_y = y/2$. *The divergence is*

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\ &= \frac{\partial}{\partial x}\left(\frac{x}{4}\right) + \frac{\partial}{\partial y}\left(\frac{y}{2}\right) \\ &= \frac{3}{4}.\end{aligned}$$

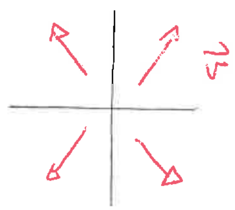
b) The vector field is as illustrated



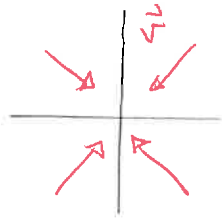
The components are $v_x = -y/2$ *and* $v_y = x/2$. *The divergence is*

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\
&= \frac{\partial}{\partial x}\left(-\frac{y}{2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{2}\right) \\
&= 0.
\end{aligned}
$$

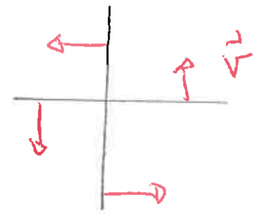
These examples illustrate the idea that the divergence describes the extent to which a field diverges away from any point



$\vec{\nabla}\cdot\vec{v} > 0$



$\vec{\nabla}\cdot\vec{v} < 0$

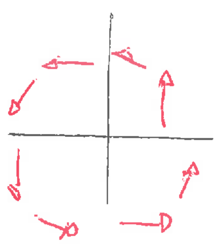


$\vec{\nabla}\cdot\vec{v} = 0$

Curl of a vector field

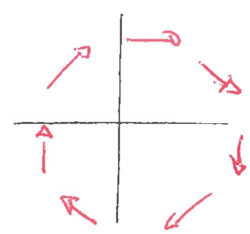
The curl is a differentiation operation which quantifies how vector fields curl around a point

$$\vec{v} = -\frac{y}{z}\hat{x} + \frac{x}{z}\hat{y}$$



Counterclockwise

$$\vec{v} = \frac{y}{z}\hat{x} - \frac{x}{z}\hat{y}$$



clockwise

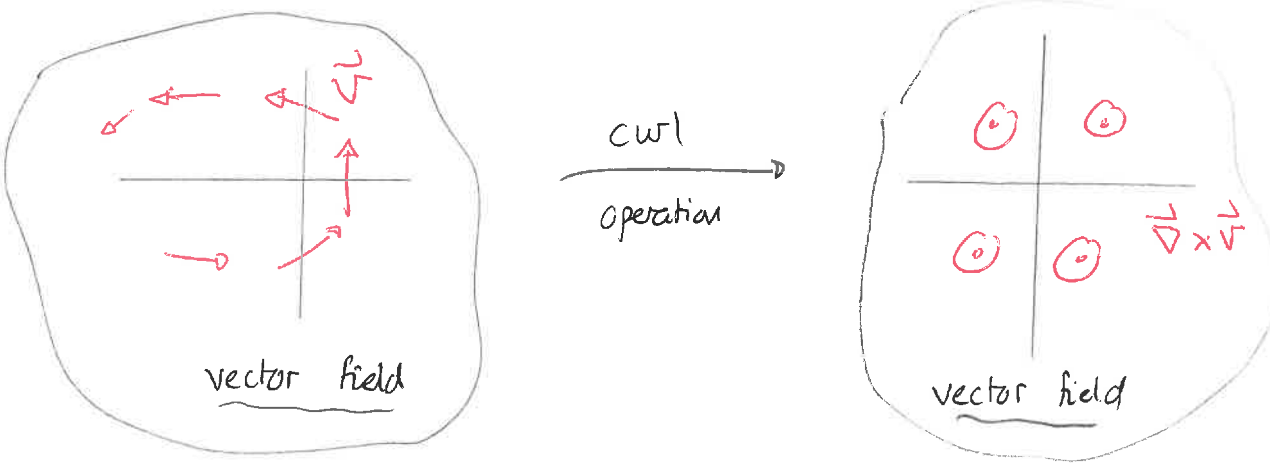
Suppose that, in Cartesian co-ordinates

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

Then the curl of $\vec{v}$ is

$$\vec{\nabla}\times\vec{v} := \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

So the curl maps:



We will see that

$\Rightarrow \vec{\nabla}\times\vec{v}$ is out $\qquad \Rightarrow \vec{\nabla}\times\vec{v}$ is in

Notes:

1) The curl does not depend on the choice of Cartesian basis
2) $\vec{\nabla}\times\vec{v}$ is a vector and must include unit vectors
3) $\vec{\nabla}\times(\vec{u}+\vec{v}) = \vec{\nabla}\times\vec{u} + \vec{\nabla}\times\vec{v}$
4) $\vec{\nabla}\times\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\times\left(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}\right)$

$$= \hat{x}\frac{\partial}{\partial x}\times v_x\hat{x} + \hat{x}\frac{\partial}{\partial x}\times v_y\hat{y} + \hat{x}\frac{\partial}{\partial x}\times v_z\hat{z} + \dots$$

$$= \frac{\partial v_x}{\partial x}\cancel{(\hat{x}\times\hat{x})}^{0} + \frac{\partial v_y}{\partial x}\underbrace{(\hat{x}\times\hat{y})}_{\hat{z}} + \frac{\partial v_z}{\partial x}\underbrace{(\hat{x}\times\hat{z})}_{-\hat{y}} + \dots$$

$$= \frac{\partial v_y}{\partial x}\hat{z} - \frac{\partial v_z}{\partial x}\hat{y} + \dots$$

3 Curl examples

Determine the curl of:

a) $\mathbf{v} = \frac{x}{4}\,\hat{\mathbf{x}} + \frac{y}{2}\,\hat{\mathbf{y}}$

b) $\mathbf{v} = \frac{-y}{2}\,\hat{\mathbf{x}} + \frac{x}{2}\,\hat{\mathbf{y}}$

Answer:

a) Here

$$\begin{aligned} v_x &= \frac{x}{4} \\ v_y &= \frac{y}{2} \end{aligned}$$

Thus

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{4} & \frac{y}{2} & 0 \end{vmatrix} \\ &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{y}{2}\right)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}\left(\frac{x}{4}\right) - \frac{\partial}{\partial x}(0)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{y}{2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{4}\right)\right] \\ &= \hat{\mathbf{x}}\,[0-0] + \hat{\mathbf{y}}\,[0-0] + \hat{\mathbf{z}}\,[0-0] \\ &= 0. \end{aligned}$$

b) Here

$$\begin{aligned} v_x &= -\frac{y}{2} \\ v_y &= \frac{x}{2} \end{aligned}$$

Thus

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{2} & \frac{x}{2} & 0 \end{vmatrix} \\ &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{2}\right)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}\left(\frac{-y}{2}\right) - \frac{\partial}{\partial x}(0)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{2}\right)\right] \\ &= \hat{\mathbf{x}}\,[0-0] + \hat{\mathbf{y}}\,[0-0] + \hat{\mathbf{z}}\left[\frac{1}{2} + \frac{1}{2}\right] \\ &= \hat{\mathbf{z}}. \end{aligned}$$