Phys 311 2020

Lecture 3

Fri: HW by 5pm

Mon: Read 1.2.3, 1.2.4

Functions in three dimensions

Electromagnetic theory uses quantities that depend on position, in other words functions in three dimensions. Examples are:

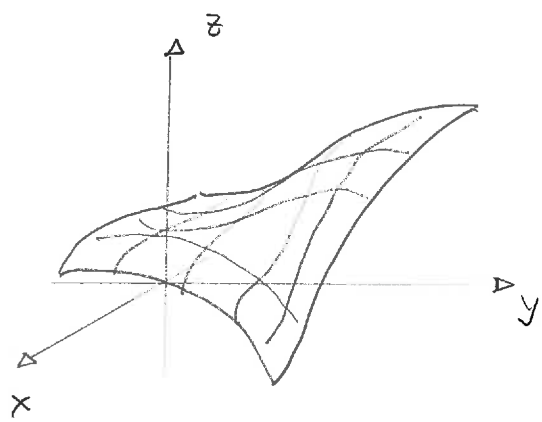
1) Electric fields
2) Magnetic fields

→ one vector at each location

3) Electrostatic potentials → one value at each location

These functions will generally vary from one location to neighboring locations and electromagnetic theory will consider such variations. This amounts to a calculus for three dimensional functions. There will be two important classes of functions

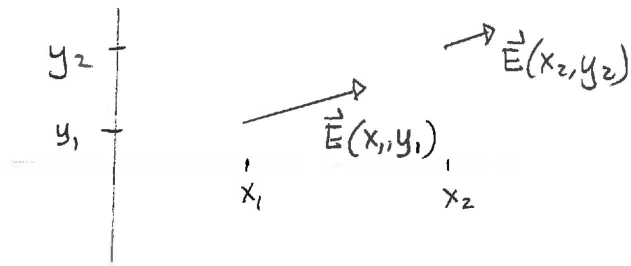
1) scalar functions - at each location the function returns a single number. For example the electrostatic potential due to a point charge at the origin.



location function

$$(x,y,z) \xrightarrow{V} V(x,y,z) = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{q}{(x^2+y^2+z^2)^{1/2}}}_{\text{given } x,y,z \text{ returns a single number}}$$

2) <u>vector functions</u> - at each location the function returns a single <u>vector</u>
An example is the electric field due to a point charge at the origin



location

$$(x,y,z) \xrightarrow{\text{function}} \vec{E} = \vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0} \frac{q}{(x^2+y^2+z^2)^{3/2}} \underbrace{(x\hat{x} + y\hat{y} + z\hat{z})}_{\text{vector}}$$

given a choice of x, y, z

number

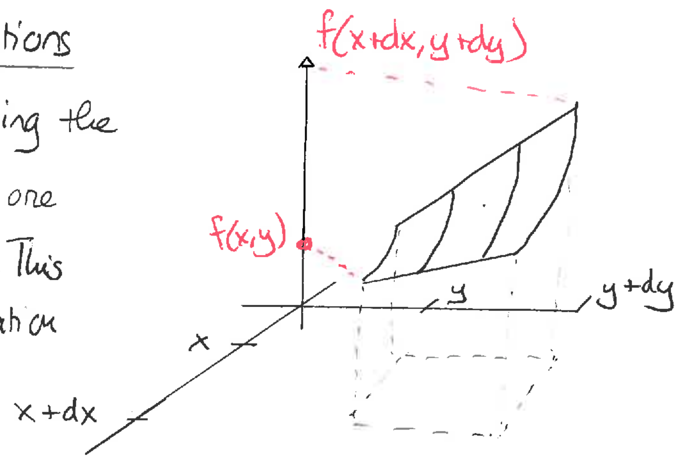
number

Vector functions are also called vector fields.

We will need a formalism for differentiating and integrating both types of function

Differentiation of scalar functions

Differentiation involves determining the change in a function from one location to a nearby location. This depends on where the nearby location is relative to the original location



For a function, such as $f(x,y) = \alpha\, y/x$, the change is different when only x varies versus when only y varies. We need a scheme for describing all possible changes

The conceptual idea is:

"Initial" location

(x, y, z)

evaluate function

$f \to f(x,y,z)$

"Final" location

$(x+dx,\ y+dy,\ z+dz)$

evaluation function

$f \to f(x+dx,\ y+dy,\ z+dz)$

calculus implies

$$f(x+dx,\ y+dy,\ z+dz) = f(x,y,z) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

evaluate at (x,y,z)

Change in f:

$$df = f(x+dx,\ y+dy,\ z+dz) - f(x,y,z)$$

$$\Rightarrow \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Thus we arrive at an expression for the change in f, starting at a point (x,y,z). It allows for us to choose dx, dy, dz independently and thus explore changes in all possible directions. We only need the three partial derivatives each evaluated at x,y,z. The direction in which we explore can be described by the vector

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

The change is described by the three partial derivatives that can also be arranged into a vector called the gradient of f:

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}}$$

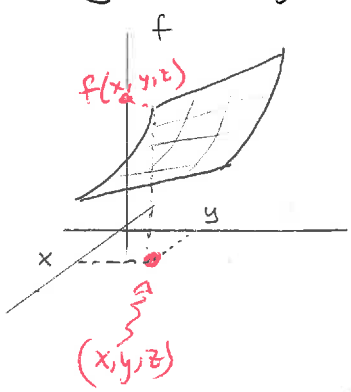
and thus (as $\vec{dl} \to 0$)

$$\boxed{df = \vec{\nabla} f \cdot \vec{dl}}$$

Conceptually

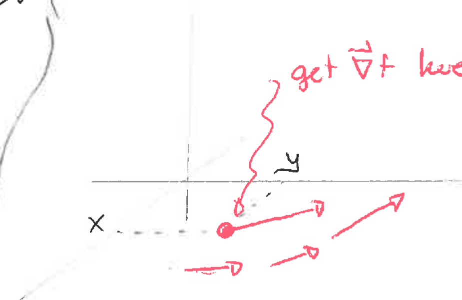
Given a function f

we want a change starting at x, y, z



e.g. $f = \alpha \frac{y}{x}$

Construct the gradient of f – one vector at each location



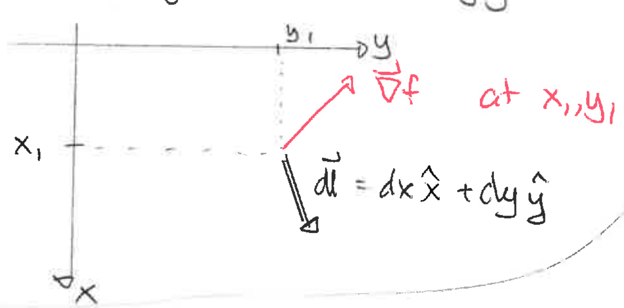
e.g. $f = \alpha \frac{y}{x} \Rightarrow$

$\Rightarrow \vec{\nabla} f = -\alpha \frac{y}{x^2}\hat{x} + \alpha \frac{1}{x}\hat{y}$

To find change in f at a location, say (x_1, y_1, z_1)

1) evaluate $\vec{\nabla} f$ at x_1, y_1, z_1

2) get direction of change $\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$



Then

$df \approx \vec{\nabla} f \cdot \vec{dl}$

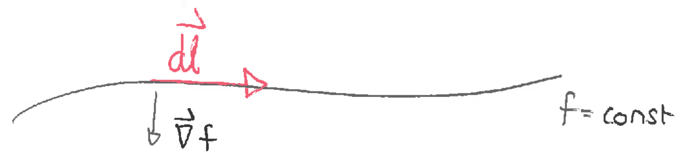
Several results can be proved

1) <u>direction of gradient</u>

> The gradient of f, $\vec{\nabla} f$, is perpendicular to contours on which f is constant. It points in the direction of increasing f.

To show this consider a contour along which f is constant. Let $\vec{dl}$ be tangent to the contour.

So



$$df = 0$$

$$\Rightarrow \quad \vec{\nabla} f \cdot \vec{dl} = 0 \quad \Rightarrow \quad \vec{\nabla} f \text{ perpendicular to contour. Now}$$

$$\vec{\nabla} f \cdot \vec{dl} = df = |dl| |\vec{\nabla} f| \underbrace{\cos\theta}_{\theta = 0} \quad \Rightarrow \quad df > 0$$

$\theta = 0 \rightarrow$ aligned in increasing f.

2) <u>maximum rate of change of f</u>

> The maximum rate of change of f is along $\vec{\nabla} f$ and has magnitude $|\vec{\nabla} f|$.

To prove this:

$$\vec{\nabla} f \cdot \vec{dl} = |\vec{\nabla} f| |\vec{dl}| \cos\theta = df$$

Thus
$$\frac{df}{|\vec{dl}|} = |\vec{\nabla} f| \cos\theta$$

is max when $\theta = 0°$ and gives $|\vec{\nabla} f|$

1 Gradients

Let

$$f(x,y) = \frac{1}{2}xy.$$

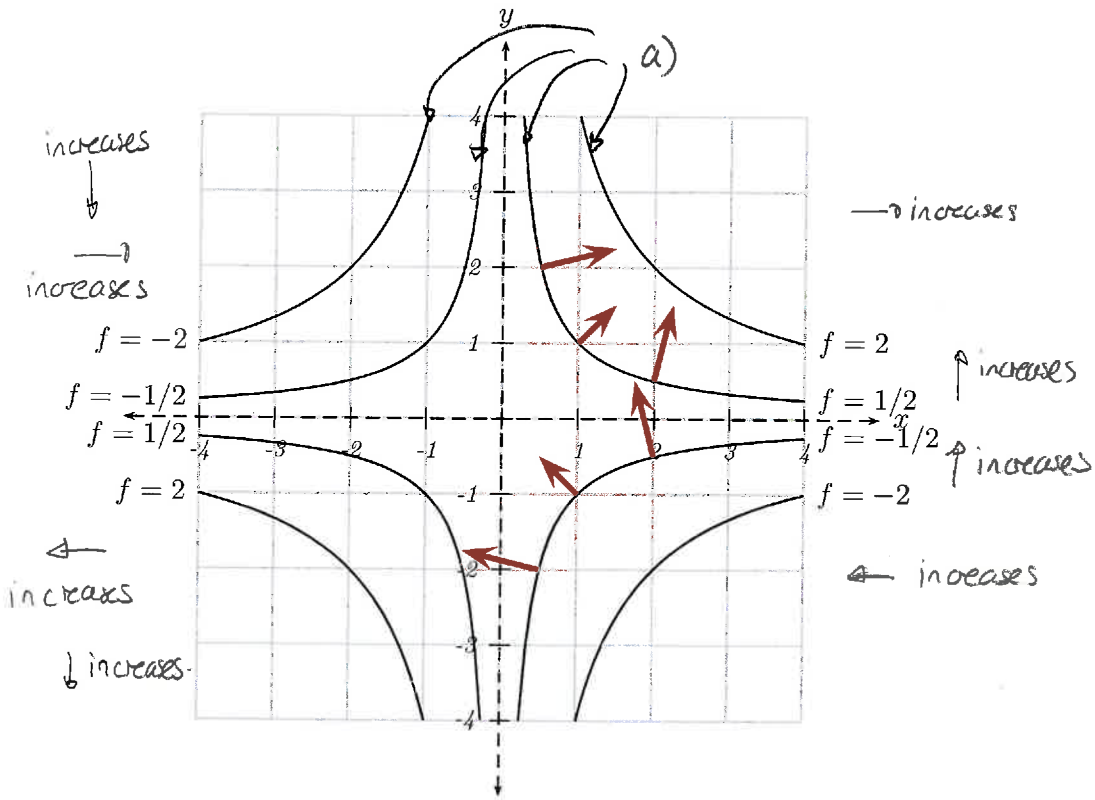
be a function in two dimensions.

a) Sketch the contours of f for which $f(x,y) = 1/2$, $f(x,y) = 2$, $f(x,y) = -1/2$, and $f(x,y) = -2$. Indicate the directions in which $f(x,y)$ increases and decreases in all four quadrants.

b) Determine ∇f. Sketch the resulting vectors along the $f(x,y) = 1/2$ contour at points such as $(1,1)$, $(2,1/2)$, ... Repeat this for the $f(x,y) = -1/2$ contour. Are these consistent with the directions in which the function increases or decreases?

Answer:

a), b)



b) Yes consistent