Phys 311 2011

Lecture 2

Fri: HW due by 5pm

~~[illegible]~~ ↳ submit to D2L

→ Course Content → Drop Box → Homework Assignment Drop Box.

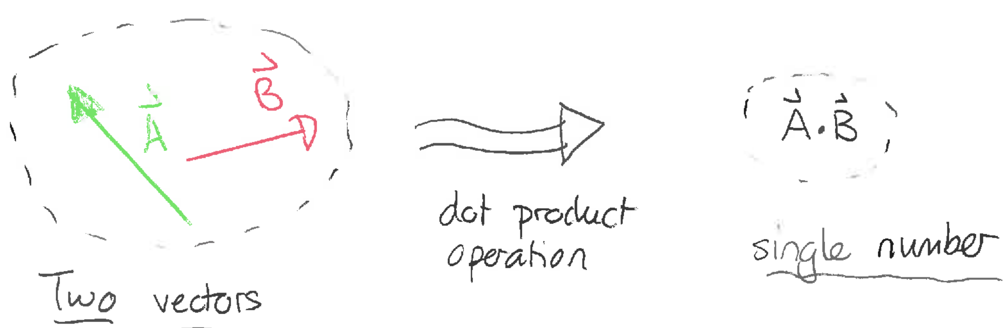
↳ late HW policy 2% per hour.

↳ format - pdf - scanned or word

Fri: Read 1.2.1, 1.2.2

Vector Dot Product

Every set of vectors or vector space always comes with the operations of vector addition and scalar multiplication. However some vector spaces can be equipped with additional algebraic operations. One example is the vector dot product (inner product). This maps two vectors to a scalar



There are various ways in which the dot product can be defined.

Suppose that $\vec{A}$ and $\vec{B}$ are two vectors which are represented in a Cartesian basis as

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

Then the dot product of $\vec{A}$ and $\vec{B}$ is

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Starting with the definition, it is straightforward to show that the dot product satisfies:

1) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

2) $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

3) $(\lambda \vec{A}) \cdot \vec{B} = \lambda \vec{A} \cdot \vec{B}$

4) $\vec{A} \cdot \vec{A} \geq 0$ with $\vec{A} \cdot \vec{A} = 0 \Leftrightarrow \vec{A} = 0$

We can use the basic definition to show that

$$\hat{x} \cdot \hat{x} = (1\hat{x} + 0\hat{y} + 0\hat{z}) \cdot (1\hat{x} + 0\hat{y} + 0\hat{z}) = 1$$

and all possibilities are

$\hat{x} \cdot \hat{x} = 1$	$\hat{x} \cdot \hat{y} = 0$
$\hat{y} \cdot \hat{y} = 1$	$\hat{y} \cdot \hat{z} = 0$
$\hat{z} \cdot \hat{z} = 1$	$\hat{z} \cdot \hat{x} = 0$

We can start with these and compute dot products purely algebraically

$$\vec{A}\cdot\vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot B_x\hat{x}$$

$$+ (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot B_y\hat{y}$$

$$+ (\quad \cdots\cdots \quad)\cdot B_z\hat{z}$$

$$= A_xB_x\,\overbrace{\hat{x}\cdot\hat{x}}^{1} + A_yB_x\,\overbrace{\hat{y}\cdot\hat{x}}^{0} + A_zB_x\,\overbrace{\hat{z}\cdot\hat{x}}^{0}$$

$$+ A_xB_y\,\underbrace{\hat{x}\cdot\hat{y}}_{0} + A_yB_y\,\underbrace{\hat{y}\cdot\hat{y}}_{1} + A_zB_y\,\underbrace{\hat{z}\cdot\hat{y}}_{0}$$

$$+ \cdots$$

$$= A_xB_x + A_yB_y + A_zB_z$$

Once we have a dot product we can define the magnitude of a vector via:

$$\boxed{\begin{array}{l}\text{Given vector } \vec{A}\text{, the magnitude of } A \text{ is} \\ A = |\vec{A}| := \sqrt{\vec{A}\cdot\vec{A}}\end{array}}$$

Then a general theorem from linear algebra (Cauchy-Schwarz inequality) states that

$$-1 \le \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} \le 1$$

so we can define an angle between two vectors via

$$\boxed{\begin{array}{l}\text{If } \vec{A},\vec{B} \text{ are two vectors then the angle between them, } \theta\text{, satisfies} \\ \cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}\end{array}}$$

Thus:

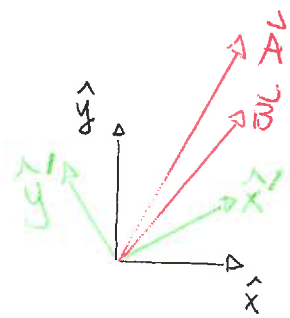
$$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta}$$

We can also define orthogonality associated with $\theta = 90°$ as

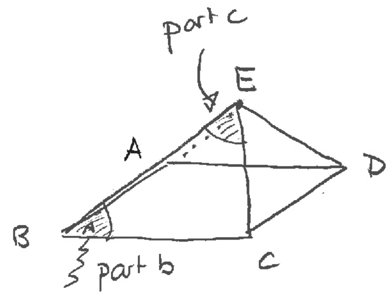
$$\boxed{\vec{A} \text{ and } \vec{B} \text{ are orthogonal} \iff \vec{A}\cdot\vec{B} = 0}$$

A remaining important piece of business concerns the inner product as calculated using two different bases.

The issue is that any vector $\vec{A}$ will have different components in different bases. These components will enter into the dot product calculation.



We can show that if the two bases are related via a rotation then the eventually outcome of the dot product calculation is the same regardless of the basis used



| 2 **Pyramid face angles**

A pyramid has a square base with sides of length L. The apex of the pyramid is a height h above the center of the base. The base lies in the first quadrant of the xy plane. Let A be the corner at the origin, B that along the x axis, C that away from either axis and D that along the y axis. Let E be the apex.

a) Determine an expression for the length of the side from B to E.

b) Determine the angle on the EBC face at point B.

c) Determine the angle on the EBC face at point E.

Answer: a) This is

$$r_{B\,to\,E} = \sqrt{\vec{r}_{B\,to\,E} \cdot \vec{r}_{B\,to\,E}}$$

Here

$$\vec{r}_{B\,to\,E} = -\frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} + h\hat{z}$$

$$= \sqrt{\left(-\frac{L}{2}\right)^2 + \left(-\frac{L}{2}\right)^2 + h^2}$$

$$r_{B\,to\,E} = \sqrt{L^2/2 + h^2}$$

b) $$\cos\theta = \frac{\vec{r}_{B\,to\,E} \cdot \vec{r}_{B\,to\,C}}{r_{B\,to\,E}\; r_{B\,to\,C}}$$

where $\vec{r}_{C\,to\,E} = -\frac{L}{2}\hat{x} - \frac{L}{2}\hat{y} + h\hat{z}$

and $$\vec{r}_{B\,to\,E} \cdot \vec{r}_{B\,to\,C} = \left(-\frac{L}{2}\right)0 + \left(+\frac{L}{2}\right)L + h\,0$$

$$= L^2/2$$

Thus $$\cos\theta = \frac{L^2/2}{\sqrt{L^2/2 + h^2}\; L} = \frac{\cancel{L}/2}{\frac{\cancel{L}}{\sqrt{2}}\sqrt{1 + h^2 2/L^2}}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}\sqrt{1 + 2h^2/L^2}}$$

This ranges from $\theta = 45^\circ\ (h = 0)$ to $\theta = 90^\circ\ (h = \infty)$

c) Here

$$\cos\theta = \frac{\vec{r}_{E\text{ to }B} \cdot \vec{r}_{E\text{ to }C}}{r_{E\text{ to }B}\; r_{E\text{ to }C}}$$ ← each is $\sqrt{L^2/2 + h^2}$

$$= \frac{(-\vec{r}_{B\text{ to }E})\cdot(-\vec{r}_{C\text{ to }E})}{\left(\sqrt{L^2/2 + h^2}\right)^2} = \frac{\vec{r}_{B\text{ to }E}\cdot\vec{r}_{C\text{ to }E}}{L^2/2 + h^2}$$

$$= \frac{1}{L^2/2 + h^2}\left\{\left(-\frac{L}{2}\right)\left(-\frac{L}{2}\right) + \left(\frac{L}{2}\right)\left(-\frac{L}{2}\right) + h\,h\right\}$$

$$= \frac{h^2}{h^2 + L^2/2}$$

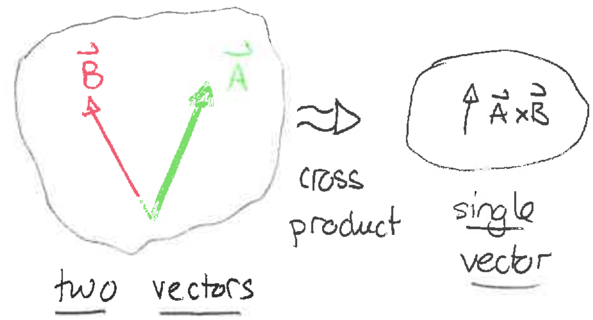
$$\Rightarrow \cos\theta = \frac{1}{1 + \frac{L^2}{2h^2}}$$

This ranges from $\theta = 0°$ $(h = \infty)$ to $\theta = 45°$ $(h \to 0)$

Vector Cross Product

The vector cross product is a multiplication operation that maps two vectors to another vector.

This only exists in three dimensional spaces. We can define this as follows:



If $\vec{A},\vec{B}$ can be expressed in Cartesian bases as

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

then

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$

Then tedious but straightforward mathematics shows that the same vector results regardless of which Cartesian basis is used. We can also calculate the cross product via:

determinant

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

where a determinant is computed by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

and here

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma$$

One can show that the cross product satisfies:

$$\boxed{\begin{array}{ll} 1)\ \vec{A}\times\vec{B} = -\vec{B}\times\vec{A} & \text{not commutative} \\ 2)\ \vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C} & \\ 3)\ (\lambda\vec{A})\times\vec{B} = \lambda(\vec{A}\times\vec{B}) & \\ 4)\ \vec{A}\times\vec{A} = 0 & \end{array}}$$

For Cartesian basis vectors:

$$\boxed{\begin{array}{llll} \hat{x}\times\hat{x} = 0 & \hat{x}\times\hat{y} = \hat{z} & \hat{y}\times\hat{z} = \hat{x} & \hat{z}\times\hat{x} = \hat{y} \\ \hat{y}\times\hat{y} = 0 & & & \\ \hat{z}\times\hat{z} = 0 & & & \end{array}}$$

2 Cross products of basis vectors

Starting with the definition of the cross product

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\,\hat{\mathbf{x}} + (A_z B_x - A_x B_z)\,\hat{\mathbf{y}} + (A_x B_y - A_y B_x)\,\hat{\mathbf{z}}$$

show that

a) $\hat{\mathbf{x}} \times \hat{\mathbf{x}} = 0$,

b) $\hat{\mathbf{y}} \times \hat{\mathbf{y}} = 0$,

c) $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, and

d) $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$.

Answer:

a) $\vec{A} = \hat{x} = \overset{A_x}{1}\hat{x} + 0\hat{y} + 0\hat{z}$

$\vec{B} = \hat{x} = \underset{B_x}{1}\hat{x} + 0\hat{y} + 0\hat{z}$

$$\vec{A} \times \vec{B} = (\underset{0}{\cancel{A_y}} B_z - \underset{0}{\cancel{A_z}} B_y)\hat{x} + (\underset{0}{\cancel{A_z}} B_x - A_x \underset{0}{\cancel{B_z}})\hat{y} + (A_x \overset{0}{\cancel{B_y}} - \overset{0}{\cancel{A_y}} B_x)\hat{z}$$

$= 0 \qquad \Rightarrow \quad \hat{x} \times \hat{x} = 0$

b) similar to a)

c) $\vec{A} = \hat{x} = 1\hat{x} + 0\hat{y} + 0\hat{z}$

$\vec{B} = \hat{y} = 0\hat{x} + 1\hat{y} + 0\hat{z}$

$$\vec{A} \times \vec{B} = (\cancel{A_y} B_z - \cancel{A_z} B_y)\hat{x} + (\overset{0}{\cancel{A_z}} B_x - A_x \overset{0}{\cancel{B_z}})\hat{y} + (\underbrace{A_x B_y}_{1} - \overset{0}{\cancel{A_y}} \cancel{B_x})\hat{z} = \hat{z}$$

d) $\vec{A} = \hat{y} = 0\hat{x} + 1\hat{y} + 0\hat{z}$

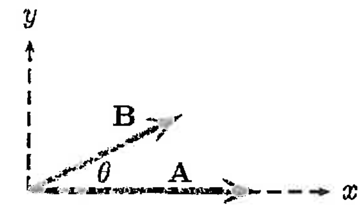
$\vec{B} = \hat{z} = 0\hat{x} + 0\hat{y} + 1\hat{z}$

$$\vec{A} \times \vec{B} = (1 - 0)\hat{x} + (0 - 0)\hat{y} + (0 - 0)\hat{z}$$

$$= \hat{x}$$

2 Geometry of the cross product

Two vectors in the xy plane are illustrated. Express **B** in the standard basis using B and θ and use this to determine an expression for $\mathbf{A} \times \mathbf{B}$.



Answer:

$$\vec{B} = B\cos\theta\,\hat{x} + B\sin\theta\,\hat{y}$$

$$\vec{A} = A\hat{x}$$

So $$\vec{A} \times \vec{B} = A\hat{x} \times (B\cos\theta\,\hat{x} + B\sin\theta\,\hat{y})$$

$$= AB\cos\theta\,\underbrace{\cancel{\hat{x}\times\hat{x}}}_{0} + AB\sin\theta\,\underbrace{\hat{x}\times\hat{y}}_{\hat{z}}$$

$$= AB\sin\theta\,\hat{z}$$

Triple Products 1.1.3 useful! beware!

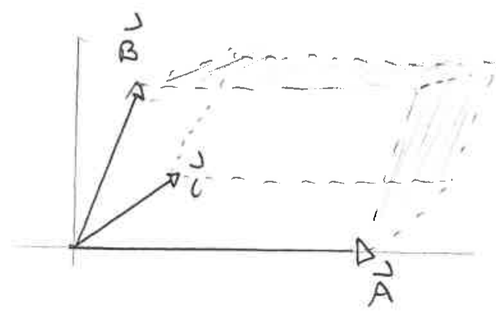
Various triple products exist and appear in electromagnetism.

1) <u>scalar triple product</u>

Given three vectors $\vec{A}, \vec{B}, \vec{C}$ the quantity $\vec{A}\cdot(\vec{B}\times\vec{C})$ is a scalar. One can show:

a) $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

b) the volume of a parallel-sided box made by $\vec{A}, \vec{B}, \vec{C}$ is given by the triple product



2) <u>vector triple product</u>

This takes three vectors $\vec{A}, \vec{B}, \vec{C}$ and produces $\vec{A}\times(\vec{B}\times\vec{C})$. One can show

a) In general $\underbrace{\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}}_{\text{Beware}}$ although there are exceptions

b) $$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$