Electromagnetic Theory: Homework 22

Due: 10 November 2020

1 Continuity equation: non-uniform density

An infinitely long cylinder of radius R carries fixed charges. The charge density at $t = 0$, given in cylindrical coordinates, is

$$\rho(\mathbf{r}) = \rho_0 e^{-z^2/a^2}$$

where a is a constant with units of length and ρ_0 has units of charge per volume. The cylinder is dragged with speed v long its axis.

a) Sketch a plot of the charge density as a function of z at $t = 0$. At which value of z is the density greatest?
b) Sketch a plot of the charge density as a function of z at $t = 1$. At which value of z is the density greatest?
c) Show that the density at any later time is

$$\rho(\mathbf{r}, t) = \rho_0 e^{-(z-vt)^2/a^2}.$$

d) Determine an expression for the current density at any time and show that it satisfies the continuity equation.

2 Current in Ohmic conductors

Show that for any Ohmic conductor, the total current through any surface is

$$I = \sigma \int_{\text{Surface}} \mathbf{E} \cdot d\mathbf{a}.$$

where σ is the conductivity.

3 Current in an Ohmic medium between two spheres

The space between two concentric conducting spherical shells is filled with an Ohmic medium with conductivity σ. The inner sphere has radius a and is at potential $V(a)$ and the outer sphere has radius b and is at potential $V(b) < V(a)$.

a) Show that between the spheres the potential is

$$V(r) = \frac{V(b) - V(a)}{b - a}\, b \left[1 - \frac{a}{r}\right] + V(a).$$

Prove this by showing that V satisfies the Poisson equation in the region between the spheres and that it gives the correct results on the boundaries of this region.

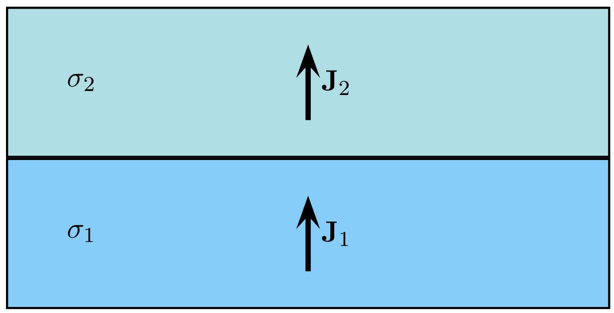
b) Determine the electric field between the spheres and the current density between the spheres.

c) Determine the resistance of the arrangement.

d) Find the resistance of the arrangement in the limit as $b \gg a$.

4 Current flowing through media with variable conductivities

Consider two infinite rectangular slabs of Ohmic material and a current that flows perpendicular to the boundary between them. The two slabs have conductivities given by σ_1 and σ_2.



The current densities in the slabs are indicated. This exercise will use the matching conditions for electric fields and potentials above and below the boundary between two media. The electric fields satisfy

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\alpha}{\epsilon_0}\hat{\mathbf{n}}$$

where here α is the surface charge density on the boundary between the media and $\hat{\mathbf{n}}$ is the normal vector from the surface below to the surface above.

a) Describe why, in a steady state, the two current densities are equal.

b) Use the current densities to determine an expression for $\mathbf{E}_2 - \mathbf{E}_1$ (where these are the electric fields in the two media) in terms of the current density in the lower material. Show that this implies that there is a non-zero charge density between the two surfaces and determine an expression for this charge density.

c) Given a patch in the boundary of surface area A, determine an expression for the charge on this surface in terms of the current I that flows through the surface.

d) Use the result from the previous part to relate the charge at the boundary to the current for the case where the lower material is a perfect conductor. Suppose that material 2 were sandwiched between two conductors. Determine a relationship between the capacitance of the arrangement and the resistance of material 2.