Electromagnetic Theory: Class Exam II

13 November 2020

Name: Solution **Total:** **/50**

Instructions

- There are 5 questions on 6 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

Integrals

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

In the following question do **either part a) or part b) for full credit.** If you do both parts, each will be graded and you will be given the highest score that you obtained for one of the parts.

a) A spherical conductor has radius R and the surface of the conductor is at potential V_0. There is no charge in the region beyond the sphere and the potential in this region only depends on r. Solve Laplace's equation to determine the potential at all points beyond the sphere (i.e. $r > R$) and find the unique potential that matches the potential on the conductor when $r = R$.

$\nabla^2 V = 0$ in this region

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \underbrace{\frac{\partial V}{\partial \theta}}_{=0}\right) + \frac{1}{r^2 \sin^2\theta}\underbrace{\frac{\partial^2 V}{\partial \phi^2}}_{=0} = 0$$

V only depends on r $\Rightarrow$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow \quad r^2 \frac{\partial V}{\partial r} = \alpha \quad = \text{const}$$

$$\Rightarrow \quad \frac{\partial V}{\partial r} = \frac{\alpha}{r^2}$$

$$\Rightarrow \quad V = -\frac{\alpha}{r} + \beta \qquad \beta = \text{const}$$

When $r = R$ $\quad V = V_0 \Rightarrow \quad V_0 = -\frac{\alpha}{R} + \beta \quad \Rightarrow \beta = V_0 + \frac{\alpha}{R}$

So $\quad V = \frac{\alpha}{R} - \frac{\alpha}{r} + V_0$

$$V = -\frac{\alpha}{r} + \frac{\alpha}{R} + V_0$$

As $r \to \infty$ $V \to 0$ $\Rightarrow \frac{\alpha}{R} = -V_0$

$\Rightarrow \alpha = -V_0 R$ $\quad$ So $\boxed{V = \frac{V_0 R}{r}}$

Question 1 continued ...

b) A sphere of radius R contains charge whose density in spherical coordinates is $\rho(\mathbf{r}') = \alpha\cos(\theta')$ where $\alpha > 0$ is a constant with units of $\mathrm{C/m^3}$. Determine the electric dipole moment of the charge distribution. Hint: consider the symmetry of the charge distribution.

$$\vec{p} = \int \vec{r}'\rho(\vec{r}')\,d\tau' \qquad +1$$

$$\left.\begin{array}{l} 0 \le r' \le R \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\} \quad d\tau' = r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'$$

$$\vec{p} = \alpha\int_0^R dr'\int_0^{2\pi} d\phi'\int_0^{\pi} d\theta'\, r'^2\sin\theta'\, \overbrace{(r'\hat{r})}^{\vec{r}'}\cos\theta' \quad \Big\} +3$$

$$= \alpha\int_0^R r'^3 dr'\int_0^{2\pi} d\phi'\int_0^{\pi} d\theta'\cos\theta'\sin\theta'\,\hat{r}$$

$$\hat{r} = \sin\theta'\cos\phi'\,\hat{x} + \sin\theta'\sin\phi'\,\hat{y} + \cos\theta'\,\hat{z} \qquad +3$$

The two integrals over ϕ' of $\cos\phi'$, $\sin\phi'$ integrate to zero.

Then:

$$\vec{p} = \alpha\int_0^R r'^3 dr'\int_0^{2\pi} d\phi'\int_0^{\pi}\cos^2\theta'\sin\theta'\,d\theta'\,\hat{z}$$

$$= \alpha\frac{R^4}{4}\times 2\pi\times\left[-\frac{1}{3}\cos^3\theta'\right]_0^{\pi}\hat{z} \qquad +5$$

$$= \frac{\alpha\pi R^4}{2}\left(-\frac{1}{3}\right)(-1-1)\,\hat{z} \quad \Rightarrow \quad \vec{p} = \frac{\alpha\pi R^4}{3}\hat{z}$$

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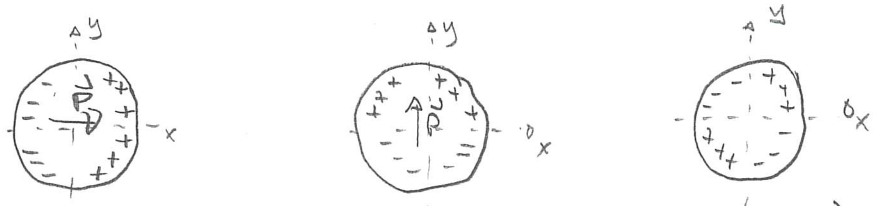
Question 2

Various cylinders, each with radius R and length L, carry charge distributions given in cylindrical coordinates via

Cylinders ~~Sphere~~ A	$\rho(\mathbf{r}') = \alpha \cos\phi'$,	
Cylinders ~~Sphere~~ B	$\rho(\mathbf{r}') = \alpha \sin\phi'$,	and
Cylinders ~~Sphere~~ C	$\rho(\mathbf{r}') = \alpha \sin 2\phi'$	

where $\alpha > 0$ is a constant. For each of these indicate the direction of the electric dipole moment. *Note: You do not have to actually calculate the dipole moment.*

A B C



$\vec{p} = p\hat{x}$ $\quad\quad$ $\vec{p} = p\hat{y}$ $\quad\quad$ $\vec{p} = 0$

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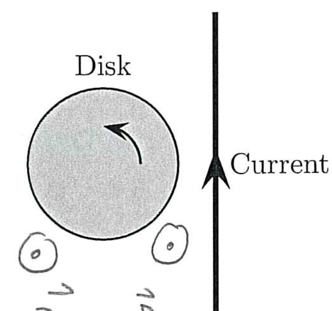
Question 3

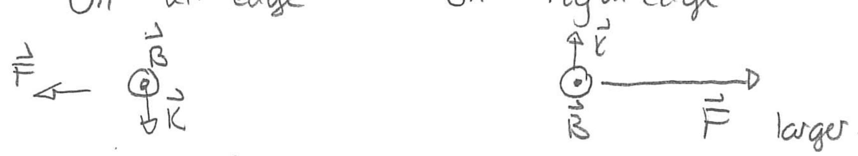
A solid disk that is uniformly charged is situated near to an infinitely long wire that carries a constant current. The disk rotates with a constant angular velocity about an axis that is perpendicular to the wire. The figure illustrates the situation. Determine the direction of the net force that the wire exerts on the disk. Explain your answer.

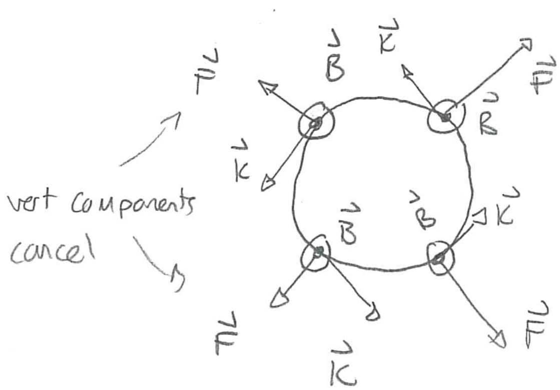


Current produces field out

$\vec{F} = \int \vec{K} \times \vec{B}\, da'$

On left edge $\quad\quad$ On right edge





net force is $\rightarrow$

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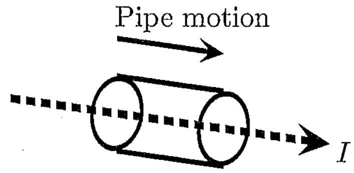
Question 4

An infinitely long cylindrical pipe with radius R carries surface charge with uniform density $\sigma > 0$. A wire runs along the axis of the pipe and carries current I. The pipe is dragged along its axis with speed v. Determine an expression for the magnetic field (at all points) produced by the entire arrangement of moving pipe and wire.



$\longrightarrow \hat{z}$

$\hookrightarrow$ surface current density in pipe

$$\vec{K} = \sigma v \, \hat{z}$$

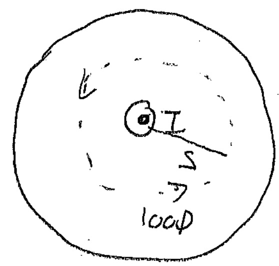
Ampère's Law $\quad \int \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$

Set axis such that $\hat{z}$ is along wire axis. Then

$$\vec{B} = B_s \hat{s} + B_\phi \hat{\phi} + B_z \hat{z}$$

By Biot-Savart $B_z = 0$. By inversion through 180° about x axis, $B_s = 0$

$$\Rightarrow \vec{B} = B_\phi(s) \hat{\phi}$$



cylinder

Use indicated loop

$$\oint \vec{B} \cdot \vec{dl} = \int B_\phi(s) \, s \, d\phi$$

$$= 2\pi s B_\phi(s) = \mu_0 I_{enc}$$

So $\quad B_\phi(s) = \dfrac{\mu_0}{2\pi} \dfrac{I_{enc}}{s}$.

If $\quad s < R \qquad I_{enc} = I$

If $\quad s > R \qquad I_{enc} = I + K \times \text{circumference}$

$$= I + 2\pi R \sigma v$$

$$\Rightarrow \vec{B} = \begin{cases} \dfrac{\mu_0}{2\pi} \dfrac{I}{s} \hat{\phi} & s < R \\[2ex] \dfrac{\mu_0}{2\pi} \left(\dfrac{I + 2\pi R \sigma v}{s} \right) \hat{\phi} & s > R \end{cases}$$

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Question 5

A magnetic vector potential is, in cylindrical coordinates, $\mathbf{A} = \alpha s^2 \hat{\phi}$ where $\alpha > 0$ is a constant.

a) Determine the magnetic field associated with $\mathbf{A}$.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \left[\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s A_\phi) - \frac{\partial A_s}{\partial \phi}\right]\hat{z}$$

$$A_s = A_z = 0 \qquad A_\phi = \alpha s^2$$

$$\Rightarrow \vec{B} = -\underbrace{\cancel{\frac{\partial}{\partial z}\alpha s^2}}_{0}\,\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}\left(s \alpha s^2\right)\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}(\alpha s^3)\hat{z}$$

$$\Rightarrow \vec{B} = \frac{1}{s} 3\alpha s^2 \hat{z} \qquad \Rightarrow \vec{B} = 3\alpha s \hat{z}$$

b) Determine the current density, $\mathbf{J}$, that produces this magnetic field and sketch this in the xy plane.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Rightarrow \vec{J} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{B}$$

$$\vec{B} = B_z \hat{z}$$

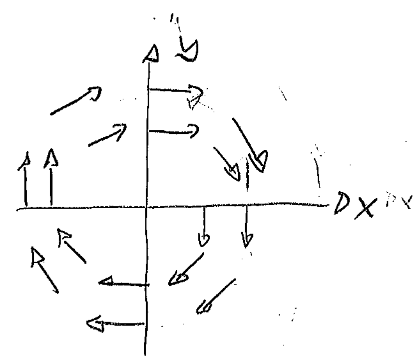
$$B_z = 3\alpha s$$

By the above

$$\vec{\nabla} \times \vec{B} = -\frac{\partial B_z}{\partial s}\hat{\phi}$$

$$= -\frac{\partial}{\partial s}(3\alpha s)\hat{\phi} = -3\alpha\hat{\phi}$$

$$\Rightarrow \vec{J} = -\frac{3\alpha}{\mu_0}\hat{\phi}$$



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