Phys 311
Fall 2020

Electromagnetic Theory: Class Exam I

2 October 2020

Name: Solution **Total:** /50

Instructions

- There are 5 questions on 6 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

Integrals

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

A solid sphere with radius R has charge density

$$\rho(\mathbf{r}') = \frac{\beta}{r'}$$

where β is a constant with units of $\mathrm{C/m^2}$. Determine an expression for the electric field at any location inside or outside of the sphere. Express your answer in terms of the total charge on the sphere.

The distribution is spherically symmetric. Thus

$$\vec{E} = E_r(r)\,\hat{r}$$

2

Use a sphere with radius r as a Gaussian surface. 1

Then $\oint \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$

For the surface
$$\left.\begin{array}{l} r' = r \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\} \quad d\vec{a} = r'^2 \sin\theta'\, d\theta'\, d\phi'\, \hat{r} = r^2 \sin\theta'\, d\theta'\, d\phi'$$

$$\oint \vec{E}\cdot d\vec{a} = \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \sin\theta'\, r^2 E_r(r) = 4\pi r^2 E_r(r)$$

4

Thus, in all cases $\quad 4\pi r^2 E_r(r) = q_{enc}/\epsilon_0$

$$E_r(r) = \frac{q_{enc}}{4\pi\epsilon_0 r^2}$$

$r < R$ Inside the sphere: $\quad q_{enc} = \int \rho\, d\tau'$

$$\left.\begin{array}{l} 0 \le r' \le r \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\} \quad d\tau' = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'$$

So $$q_{enc} = \underbrace{\int_0^r dr' r'^2 \int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\phi'}_{4\pi} \frac{\beta}{r'}$$

$$= 4\pi\beta \int_0^r r' dr' = 2\pi\beta r^2$$ 3

Thus inside $$E_r = \frac{\beta}{2\epsilon_0}$$ 1

Outside the same integral runs to $R \to$ total charge $Q = 2\pi\beta R^2$ 2

$$\beta = \frac{Q}{2\pi R^2}$$

So outside $$E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$ 1

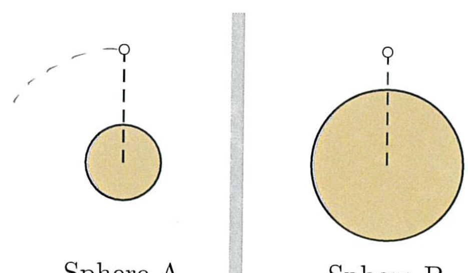
and inside $$E_r = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 R^2} & r < R \\ \frac{Q}{4\pi\epsilon_0 r^2} & r > R \end{cases}$$ 1

/15

Question 2

Two widely separated spheres carry identical total charges that are uniformly distributed. Sphere A has radius R and sphere B has radius $2R$ where R is some constant. Consider two points (indicated by "o") that are distance $3R$ from the center of each sphere. How does the magnitude of the electric field for sphere A at this point compare (same, larger, smaller) to that for B at the same point? Explain your answer.



Sphere A | Sphere B

In both cases $\vec{E} = E_r(r)\hat{r}$. We use a Gaussian sphere with radius 3R

$$\oint \vec{E}\cdot d\vec{a} = 4\pi R^2 E_r(r) = \frac{q_{enc}}{\epsilon_0} \Rightarrow E_r(r) = \frac{1}{4\pi\epsilon_0}\frac{q_{enc}}{R^2}$$

same $q_{enc} \Rightarrow$ same E

/5

Question 3

The electric field produced by an unknown charge distribution is (in spherical coordinates)

$$\mathbf{E} = E_0\, e^{-\lambda r}\,\hat{\mathbf{r}}$$

where $\lambda > 0$ is a constant with units of m^{-1} and E_0 is a constant with units of N/C.

a) Determine the electrostatic potential at all points, assuming that $V \to 0$ as $r \to \infty$.

$$\Delta V = -\int \vec{E}\cdot d\vec{l} \qquad 1$$

$$\underline{\cancel{V(\infty)} - V(\vec{r})} = -\int_{\vec{r}}^{\infty} \vec{E}\cdot d\vec{l}$$

Then $d\vec{l} = dr\,\hat{r} \Rightarrow \vec{E}\cdot d\vec{l} = E_0 e^{-\lambda r'}$ and $r < r' < \infty$

$$-V(\vec{r}) = -\int_r^{\infty} E_0 e^{-\lambda r'}\,dr' \Rightarrow V(\vec{r}) = \frac{E_0}{-\lambda}\, e^{-\lambda r'}\Big|_r^{\infty}$$

3

$$\Rightarrow V(\vec{r}) = \frac{E_0}{\lambda} e^{-\lambda r}$$

b) A particle with charge $q > 0$ and mass m is placed at a point r_0 and is released from rest. In which direction does the particle move and what is the maximum speed that it attains?

1 $\Delta K + q\Delta V = 0$

$\Delta K = -q\Delta V$

For $\Delta K > 0$

$\Delta V < 0$

V decreases

moves radially out to ∞

2

$$K_f - \cancel{K_i}^{\,0} = -q(\cancel{V_f}^{\,0} - V_i)$$

$$\frac{1}{2} m v^2 = q\frac{E_0}{\lambda} e^{-\lambda r_0}$$

4 ~~3~~

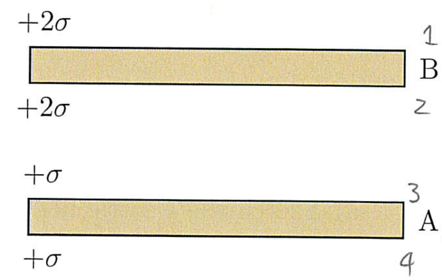
$$v^2 = \frac{2qE_0}{m\lambda} e^{-\lambda r_0}$$

$$v = \sqrt{\frac{2qE_0}{m\lambda} e^{-\lambda r_0}}$$

/12

Question 4

Two infinite slabs are parallel to each other. Each slab has two surfaces and the charge on each surface is uniformly distributed with the illustrated densities ($\sigma > 0$). There is no charge inside the slabs.



a) Describe as precisely as possible how the magnitude of the electric field inside slab A is related to that of the field inside slab B. Explain your answer.

Each sheet produces a field.

Inside B

1 2 3 4

each $\frac{\sigma}{2\epsilon_0} \Rightarrow$ total $\frac{\sigma}{\epsilon_0}$

Inside A

1 2 3 4

each $\frac{2\sigma}{2\epsilon_0} \Rightarrow \frac{2\sigma}{\epsilon_0}$ total

$\vec{E}$ Inside A <u>twice</u> $\vec{E}$ inside B.

~~+4~~ +5

b) Suppose that the separation between the slabs was increased. Would the electrostatic potential difference across the gap between the plates increase, decrease or stay constant? Explain your answer.

+3

$$|\Delta V| = \left|-\int \vec{E} \cdot d\vec{\ell}\right|$$

stay same

$$= |E| \underbrace{\int d\ell}$$

increases

$\Rightarrow \Delta V$ increases

/8

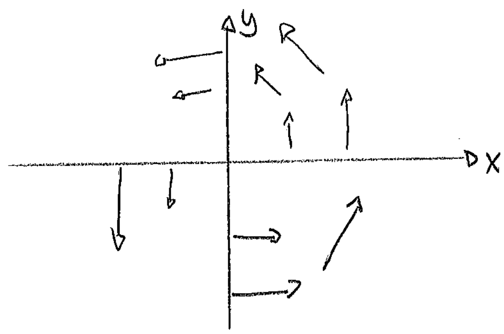
Question 5

A possible electric field has the form (in cylindrical coordinates)

$$\mathbf{E} = \frac{\alpha}{s}\hat{\phi}$$

where $\alpha > 0$.

a) Sketch the electric field in the xy plane.



b) Determine whether this is a possible electrostatic field.

Need to check $\vec{\nabla} \times \vec{E} = \left[\frac{1}{s}\cancel{\frac{\partial E_z}{\partial \phi}} - \cancel{\frac{\partial E_\phi}{\partial z}}\right]\hat{s} + \left[\cancel{\frac{\partial E_s}{\partial z}} - \cancel{\frac{\partial E_z}{\partial s}}\right]\hat{\phi}$

$$+ \frac{1}{s}\left[\frac{\partial}{\partial s}(sE_\phi) - \cancel{\frac{\partial E_s}{\partial \phi}}\right]\hat{z}$$

$E_\phi = \alpha/s$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(\cancel{s}\frac{\alpha}{\cancel{s}}\right)\hat{z}$$

$$= 0$$

So it is a possible electrostatic field.

/10