Phys 311
Fall 2019

Electromagnetic Theory: Class Exam I

4 October 2019

Name: Solution **Total:** /40

Instructions

- There are 4 questions on 6 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

Integrals

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

A sphere with radius R contains total charge that is distributed according to the charge density

$$\rho = \alpha r$$

where r is the distance from the center of the sphere and α is a constant.

4 a) Suppose that the total charge contained within the entire sphere is Q. Determine an expression for Q in terms of α and R.

$$Q = \int \rho(\vec{r}\,')\,d\tau' = \int_0^R dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta'\, r'^2 \sin\theta' \underbrace{\rho(\vec{r}\,')}_{\alpha r'}$$

$$= \alpha \int_0^R dr' r'^3 \underbrace{\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta'}_{4\pi}$$

$$Q = \frac{4\pi\alpha R^4}{4}$$

$$\Rightarrow \quad Q = \pi \alpha R^4 \qquad \alpha = \frac{Q}{\pi R^4}$$

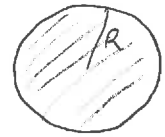
b) Determine expressions for the electric field at all points *inside* and *outside* the sphere. The expressions for the electric field must be written in terms of Q.

Use Gauss' Law $\quad \oint \vec{E} \cdot d\vec{a} = \dfrac{q_{enc}}{\epsilon_0}$ 1

1 By rotating about x, y, z axes we can see that

$$\vec{E} = E_r(r)\,\hat{r}$$

2 Choose as a Gaussian surface a sphere of radius r. On this



$$\left.\begin{array}{l} r' = r \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\} \quad d\vec{a} = r^2 \sin\theta'\, d\theta'\, d\phi'\, \hat{r}$$

Question 1 continued . . .

$$\text{So} \quad \oint \vec{E}\cdot d\vec{a} = \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \, E_r(r)\, r^2 \sin\theta'$$

$$= E_r(r)\, r^2 \underbrace{\int_0^{\pi} d\theta' \sin\theta' \int_0^{2\pi} d\phi'}_{4\pi}$$

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$$\Rightarrow \quad \oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E_r(r) = \frac{q_{enc}}{\epsilon_0}$$

$$\text{So} \quad E_r(r) = \frac{1}{4\pi\epsilon_0}\frac{q_{enc}}{r^2}$$

Then inside the sphere $(r<R)$

$$q_{enc} = \int_0^{r} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \, r'^2 \sin\theta' \, \rho(\vec{r}') = \pi \alpha r^4 \quad \text{(as in part a)}$$

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$$\text{So} \quad q_{enc} = \pi\alpha r^4 = \frac{\pi Q}{\pi}\frac{r^4}{R^4} = Q\frac{r^4}{R^2}$$

So for $r<R$ $\quad E_r = \dfrac{1}{4\pi\epsilon_0}\dfrac{Qr^2}{R^4} \Rightarrow \boxed{\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Qr^2}{R^4}\hat{r}}$

1 For $r>R$ $\quad q_{enc}=Q \quad \Rightarrow E_r = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2} \Rightarrow \boxed{\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\hat{r}}$

Q

1

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Question 2

Someone proposes the following as an electric field (given in cylindrical coordinates) produced by an arrangement of stationary charges:

$$\mathbf{E} = E\hat{\phi}$$

where E is a constant.

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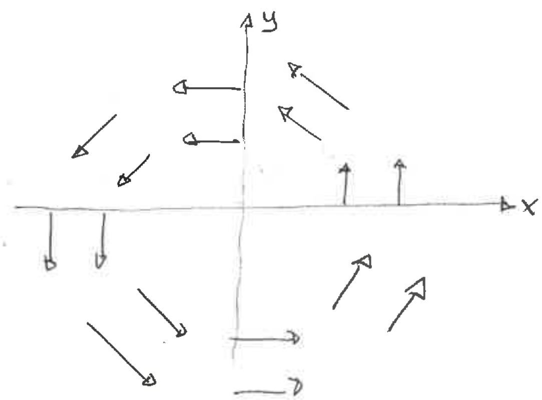
a) Sketch the electric field in the xy plane.



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b) Describe whether this electric field could arise from a collection of stationary charges or not. Explain your answer.

We need $\vec{\nabla} \times \vec{E} = 0$. Here $E_s = 0 \quad E_z = 0 \quad E_\phi = E$

In cylindrical co-ords:

$$\vec{\nabla} \times \vec{E} = \left[\frac{1}{s}\frac{\partial \cancel{E_z}^{0}}{\cancel{\partial \phi}} - \frac{\partial \cancel{E_\phi}^{0}}{\cancel{\partial z}}\right]\hat{s} + \left[\frac{\partial \cancel{E_s}^{0}}{\cancel{\partial z}} - \frac{\partial \cancel{E_z}^{0}}{\cancel{\partial s}}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s E_\phi) - \frac{\partial \cancel{E_s}^{0}}{\cancel{\partial \phi}}\right]\hat{z}$$

$$= \frac{1}{s}\frac{\partial}{\partial s} sE\,\hat{z} = \frac{E}{s}\hat{z} \neq 0$$

It does not arise from stationary charges

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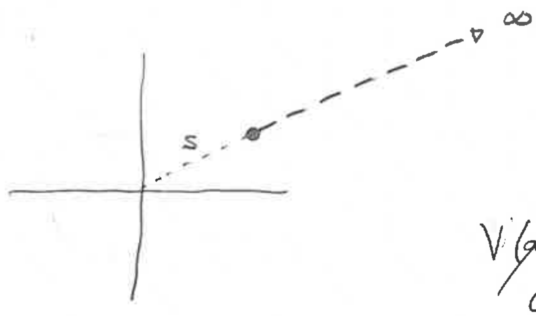
Question 3

A particular electrostatic charge distribution gives an electric field, described in cylindrical coordinates, of

$$\mathbf{E} = \frac{k}{s^2}\hat{\mathbf{s}}$$

where k is a constant. Determine the electrostatic potential at any point, taking the potential at infinity as zero.



$$\Delta V = -\int \vec{E} \cdot d\vec{l}$$ (+2)

$$\underbrace{\cancel{V(\infty)}}_{0} - V(s) = -\int_s^\infty \vec{E} \cdot d\vec{l}$$ (+1)

Here $d\vec{l} = ds'\,\hat{s} \Rightarrow \vec{E} \cdot d\vec{l} = \frac{k}{s'^2} ds'$, (+1)

$$\Rightarrow \quad -V(s) = -\int_s^\infty \frac{k}{s'^2} ds' \Rightarrow \quad V(s) = k\left(-\frac{1}{s'}\right)\Bigg|_s^\infty$$

$$= -k\left(\frac{1}{\infty} - \frac{1}{s}\right)$$

$$\Rightarrow \quad V(s) = \frac{k}{s}$$ (+6)

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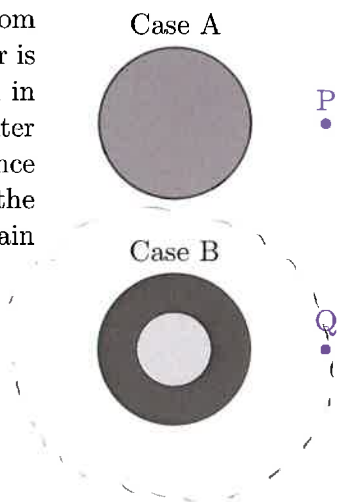
Question 4

Two infinitely long cylinders each have the same radius, R and carry charge whose distribution only depends on the radial distance from the cylinder axis. The total charge per unit length of each cylinder is identical. However, in cylinder A it is uniformly distributed and in cylinder B, the charge density increases with distance from the center of the cylinder. Consider the electric fields at points each a distance $2R$ from the cylinder axis in each case. Is the field at point Q the same as, larger than or smaller than the field at point P? Explain your answer.



Using a Gaussian cylinder with radius 2R and height h gives

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0}$$

But q_{enc} is same regardless of Case A or Case B

$\Rightarrow \vec{E}$ same

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