Electromagnetic Theory: Homework 2

Due: 27 August 2019

This assignment will be graded immediately after the due date. If you get all problems correct, then you will receive 100%. If you have made any errors, then I will deduct 10%, point the errors out and you must submit a corrected assignment by 24 August 2017. If there are still errors, then I will deduct another 10% and you must submit the corrected assignment by 31 August 2017. This will continue until you **have solved every problem correctly.**

1 Functions and Vector Fields

Consider a fluid that can flow through a three dimensional region of space. Any location in this region can be specified using coordinates x, y, z. Assume that the fluid is continuous (does not consist of individual molecules) and that the temperature, pressure and velocity of the the fluid can vary from one location to another. Describe which of the following are scalar functions (of the location x, y, z) and which are vector fields.

a) Volume of the entire fluid.
b) Mass of the entire fluid.
c) Temperature.
d) Pressure.
e) Velocity.

2 Gradient of a function

Let $V(x, y) = x^2 + y^2$.

a) Sketch several contours of $V(x, y)$ in the xy-plane. Indicate which provide larger values and which provide smaller values.
b) Determine $\boldsymbol{\nabla} V$ and sketch the resulting vector field on your contour plot.
c) Verify, using the contour sketch, that $\boldsymbol{\nabla} V$ is perpendicular to the contours.

3 Landscape

A model landscape is created above a flat two-dimensional surface; the locations on this surface are describe by coordinates x, y. The vertical height above this is given by

$$h(x, y) = -2xy - x^2 + 6x - 2y^2 + 8y + 5.$$

a) Determine the location at which the landscape is highest.
b) Determine the maximum height of the landscape.

c) Determine the direction of steepest ascent at the origin. What is the steepest slope at that point?

4 Gradients of distances

Consider a conventional coordinate system and suppose that $\mathbf{r}$ is a position vector for the locations with coordinates (x, y, z).

a) Using the general rule that the unit vector along $\mathbf{A}$ is $\hat{\mathbf{A}} = \mathbf{A}/A$, determine an expression for $\hat{\mathbf{r}}$ in terms of x, y, z and the standard basis. Determine an expression for r in terms of x, y and z.

b) For any integer n show that $\nabla(r^n) = nr^{n-1}\hat{\mathbf{r}}$. (Note that $\hat{\mathbf{r}}$ is the unit vector along $\mathbf{r}$.)

5 Gradients of separations

Consider the standard separation vector $\boldsymbol{\mathscr{r}}$. Denote the magnitude of this by $\mathscr{r}$.

a) Determine an expression for $\mathscr{r}$ in terms of x, y, z, x', y', z'.

b) Consider the operator ∇ where the derivatives in this operator refer to unprimed coordinates x, y, y. The primed coordinates can be regarded as constants. Show that:

$$\begin{aligned} \nabla(\mathscr{r}) &= \hat{\boldsymbol{\mathscr{r}}} \\ \nabla(\mathscr{r}^2) &= 2\boldsymbol{\mathscr{r}} \\ \nabla\left(\frac{1}{\mathscr{r}}\right) &= -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \end{aligned}$$

These are important results that we will use many times throughout this course.

6 Divergence of a vector field

Consider vector fields in a two dimensional plane. These have the form:

$$\mathbf{v} = v_x(x, y)\hat{\mathbf{x}} + v_y(x, y)\hat{\mathbf{y}}$$

where $v_x(x, y)$ and $v_y(x, y)$ are two functions of x and y. Construct a vector field by choosing two such functions, which must satisfy the following: they were not used previously in class, they are not just constant functions and they are relatively simple (don't try complicated expression with trigonometric and other special functions).

a) Sketch several vectors for the vector field that you chose. Based on the appearance of the vector field, will the divergence be zero or not? Explain your answer.

b) Compute the divergence of the vector field. Was your prediction correct?