Phys 311
Fall 2019

Electromagnetic Theory: Class Exam II

15 November 2019

Name: Solution **Total:** /50

Instructions

- There are 4 questions on 7 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

Integrals

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

A spherical conductor of radius R is charged. There is no charge outside of the conductor. Which of the following (choose one) is a possible potential, given in spherical coordinates, at all points **on and beyond** the surface of the conductor?

i) $V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$

ii) $V(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$

iii) $V(r,\theta,\phi) = \cancel{\frac{1}{4\pi\epsilon_0}\frac{QR}{r^2}}$

iv) $V(r,\theta,\phi) \cancel{=} \cancel{\frac{1}{4\pi\epsilon_0}}\frac{Q\cos\phi}{r}$

v) $V(r,\theta,\phi) = \text{constant}$

Briefly explain your answer.

Beyond $\nabla^2 V = -\rho/\epsilon_0 \Rightarrow \nabla^2 V = 0$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left(\sin\theta\cancel{\frac{\partial V}{\partial\theta}}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

0 for all

iv) Clearly does not satisfy this. Nor does iii)

Both i) and v) are constant. This means no field beyond. But thats impossible for a localized charge

$\Rightarrow$ ii) works

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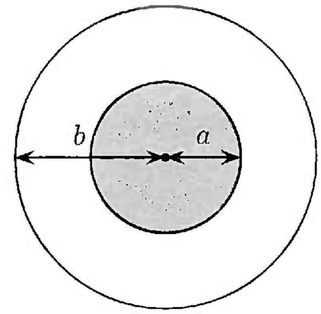
Question 2

An infinitely long cylindrical shell, of negligible thickness and radius b, surrounds an infinitely long solid cylindrical rod, of radius a. Their axes are both along the z-axis and a view down the length of this axis is illustrated. The volume current density in the rod is (in cylindrical coordinates) $\mathbf{J} = \frac{3\alpha}{2\pi a^3} s\hat{\mathbf{z}}$ where α is a constant with the dimensions of current. The current along the cylindrical shell, flows in the $-\hat{\mathbf{z}}$ direction, is uniformly distributed across the surface and is such that the total current flowing down the shell is exactly opposite to that flowing through the cylinder.



a) Show that the surface current density in the cylindrical shell is

$$\mathbf{K} = -\frac{\alpha}{2\pi b}\,\hat{\mathbf{z}}.$$

Total current in rod

$$I_{rod} = \int_{\substack{\text{rod surface}\\ \text{cross section}}} \vec{J}\cdot d\vec{a} \qquad \begin{matrix} 0 \le s \le a \\ 0 \le \phi \le 2\pi \\ z = \text{const} \end{matrix}$$

$$d\vec{a} = s\,ds\,d\phi\,\hat{z}$$

$$I_{rod} = \int_0^a ds \int_0^{2\pi} d\phi\, s^2 \frac{3\alpha}{2\pi a^3} = \frac{3\alpha}{\cancel{2\pi} a^3}\int_0^a s^2 ds \int_0^{2\pi} d\phi$$

$$= \frac{3\alpha}{3a^3} s^3\Big|_0^a = \alpha$$

Now the magnitude of $\vec{K}$ is

$$K = \frac{I_{shell}}{\text{circumference shell}} = \frac{I_{rod}}{2\pi b} = \frac{\alpha}{2\pi b}$$

Thus $\vec{K} = -\frac{\alpha}{2\pi b}\hat{z}$

Question 2 continued ...

b) Determine the magnetic field at **all locations**.

In general $\vec{B} = B_s \hat{s} + B_\phi \hat{\phi} + B_z \hat{z}$

By the Biot-Savart law $\vec{B}$ cannot have a component along $\vec{J}$ or $\vec{K}$

so $B_z = 0$

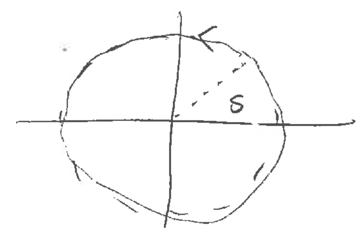
Via an inversion about any axis in the xy plane $\vec{B} \Rightarrow +B_s \hat{s} - B_\phi \hat{\phi}$

But $\vec{B} \Rightarrow -\vec{B} \Rightarrow B_s = 0$.

Then

$$\vec{B} = B_\phi(s)\,\hat{\phi}$$

Take as a loop the circle



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$2\pi s B_\phi(s) = \mu_0 I_{enc} \Rightarrow B_\phi(s) = \frac{\mu_0 I_{enc}}{2\pi s}$$

Three zones 1) $s < a$ $\quad I_{enc} = \int \vec{J} \cdot d\vec{a} = \int_0^s s' ds' \int_0^{2\pi} d\phi\, J$

$$= \frac{3\alpha}{2\pi a^3} \int_0^s s'^2 ds' \int_0^{2\pi} d\phi$$

$$= \frac{s^3}{a^3}\alpha$$

Then $\vec{B} = \frac{\mu_0 s^2}{2\pi a^3}\alpha\,\hat{\phi} \quad s < a$

2) $a < s < b \quad I_{enc} = I_{rod} = \alpha \Rightarrow \vec{B} = \frac{\mu_0 \alpha}{2\pi s}\hat{\phi}$

3) $b < s \quad I_{enc} = 0 \Rightarrow \vec{B} = 0$

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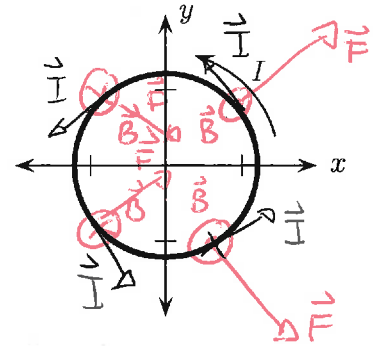
Question 3

A circular loop with radius R lies in the xy plane as illustrated and carries a current I counterclockwise. A hidden source current produces the magnetic field



$$\mathbf{B} = \alpha x \hat{\mathbf{z}}$$

where $\alpha > 0$ is a constant. Show qualitatively that the net force exerted by the field on the loop is not zero and determine the net force exerted by the field on the loop.

$$\vec{F} = \int I\, \vec{dl} \times \vec{B}$$

By the diagram $\vec{F} \neq 0$ and is in x direction.

$$\vec{dl} = R\, d\phi\, \hat{\phi}$$

$$\vec{dl} \times \vec{B} = R\, d\phi\, \alpha x\, \underbrace{\hat{\phi} \times \hat{z}}_{\hat{s}} = R\alpha x\, d\phi\, \hat{s}$$

$$= R\alpha\, R\cos\phi\, d\phi\, \hat{s} = R^2 \alpha \cos\phi\, d\phi\, \hat{s}$$

But $\hat{s} = \cos\phi\, \hat{x} + \sin\phi\, \hat{y}$

$$\Rightarrow \vec{F} = I R^2 \alpha \int_0^{2\pi} \cos\phi \left(\cos\phi\, \hat{x} + \sin\phi\, \hat{y}\right) d\phi$$

$$= I R^2 \alpha \int_0^{2\pi} \cos^2\phi\, \hat{x} + I R^2 \alpha \cancel{\int_0^{2\pi} \sin\phi \cos\phi\, d\phi\, \hat{y}}^{\,0}$$

$$\boxed{\vec{F} = I \pi R^2 \alpha\, \hat{x}}$$

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Question 4

In the following question do **either part a) or part b) for full credit.** If you do both parts, each will be graded and you will be given the highest score that you obtained for one of the parts.

a) A magnetic dipole has magnetic moment $\mathbf{m} = m\hat{\mathbf{x}}$. Determine the magnetic vector potential is spherical coordinates and use this to determine the magnetic field produced by the dipole in spherical coordinates.

$$\vec{A} = \frac{\mu_0}{4\pi r^2}\,\vec{m}\times\hat{r}$$

$$= \frac{\mu_0 m}{4\pi r^2}\,\hat{x}\times\hat{r}$$

But $\quad \hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$

$$\hat{x}\times\hat{r} = \sin\theta\cos\phi\,\cancelto{0}{\hat{r}\times\hat{r}} + \cos\theta\cos\phi\,\underbrace{\hat{\theta}\times\hat{r}}_{-\hat{\phi}} - \sin\phi\,\underbrace{\hat{\phi}\times\hat{r}}_{+\hat{\theta}}$$

$$\vec{A} = \frac{\mu_0 m}{4\pi r^2}\left[-\cos\theta\cos\phi\,\hat{\phi} - \sin\phi\,\hat{\theta}\right]$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right\}\hat{r} + \frac{1}{r}\left\{\frac{1}{\sin\theta}\cancelto{0}{\frac{\partial A_r}{\partial\phi}} - \frac{\partial}{\partial r}(rA_\phi)\right\}\hat{\phi}$$
$$+\frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\theta) - \cancel{\frac{\partial A_r}{\partial\theta}}\right\}\hat{\phi}$$

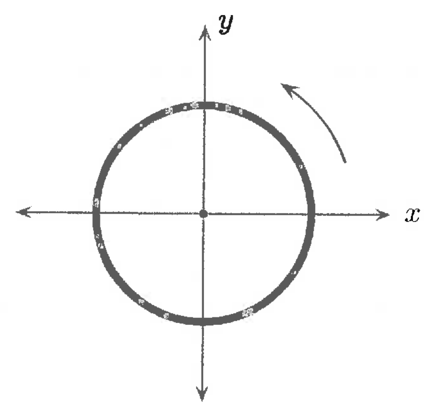
$$= \frac{-\mu_0 m}{4\pi r^2}\left[\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\cos\theta\cos\phi) - \frac{\partial}{\partial\phi}\sin\phi\right\}\hat{r}\right.$$

$$+\frac{\mu_0 m}{4\pi r}\frac{\partial}{\partial r}\left(\frac{\cos\theta\cos\phi}{r}\right)\hat{\phi} + \frac{\mu_0 m}{4\pi r}\frac{\partial}{\partial r}\frac{\sin\phi}{r}\hat{\theta}$$

$$= -\frac{\mu_0 m}{4\pi r^3}\left\{\left[\frac{\cos\phi\,(\cos^2\theta - \sin^2\theta)}{\sin\theta} - \frac{\cos\phi}{\sin\theta}\right]\hat{r}\right.$$
$$\left. - \cos\theta\cos\phi\,\hat{\phi} - \sin\phi\,\hat{\theta}\right\}$$

Question 4 continued . . .

b) A loop of radius R carries charge of uniform linear density, λ, and lies in the xy plane. The loop rotates about the axis through its center, as illustrated in the diagram, so that any point moves with speed v. Determine an expression for the magnetic dipole moment of the loop in terms of R, λ, v and constants.



For a loop

$$\vec{m} = I A \hat{n}$$

where $\hat{n}$ is perpendicular to the loop.

Here $A = \pi R^2$

$$I = \lambda v$$

$$\vec{m} = \lambda v \pi R^2 \hat{z}$$

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